1. A model for harvesting a population takes the form:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(1 - \frac{N}{K}) - Y_0$$

where $N(t)$ is the population density at time t, and r, K and Y_0 are positive constants.

   (a) Explain why this is called a constant yield model.

   (b) Show that for $Y_0 < \frac{rK}{4}$, this equation has two steady states, and explain graphically why one of these steady states is linearly stable while the other is linearly unstable.

   (c) Show that the recovery time for harvesting at a yield Y_0, $T_R(Y_0)$, satisfies

$$\frac{T_R(Y_0)}{T_R(0)} = \frac{1}{(1 - \frac{Y_0}{Y_M})^{\frac{1}{2}}}$$

   for $Y_0 < Y_M$ and $Y_M = \frac{rK}{4}$.

   (d) Why is this harvesting approach worse than the constant effort approach if we aim for maximum yield (Y_M).

   (e) Explain why this model is unrealistic for $Y_0 > Y_M$.

   (f) Why might you have anticipated right from the outset that this model could be unrealistic in certain circumstances?

2. A continuous-time model for the evolution of a prey species (density $N(t)$, where t is time), takes the form:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = RN\left(1 - \frac{N}{K}\right) - P\left(1 - \exp\left[-\frac{N^2}{\epsilon A^2}\right]\right),$$

where $0 < \epsilon \ll 1$ and R, K, P and A are positive constants.

   (a) Explain the biological interpretation of the different terms in the model.

   (b) If the units of N are density and those of t are time, what are the dimensions of R, K, P, A and ϵ? Hence show that

$$u = \frac{N}{A}, \qquad \tau = \frac{P}{A}t, \qquad r = \frac{AR}{P}, \qquad q = \frac{K}{A},$$

   are non-dimensional.

   (c) Show that the model can be non-dimensionalised to give

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = ru\left(1 - \frac{u}{q}\right) - \left(1 - \exp\left[-\frac{u^2}{\epsilon}\right]\right),$$

   where r and q are positive parameters.

   (d) Demonstrate that a necessary condition for there to be three non-zero steady states is that r and q must satisfy $rq > 4$.

   (e) Could this model exhibit hysteresis? Justify your answer.

3. Suppose that the evolution of a population can be described by a discrete-time Ricker model of the form

$$N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right],$$

where $0 < r < 2$.

(a) Describe the biological interpretation of the model.

(b) Determine any non-negative steady states and their linear stability.

(c) Construct a cobweb map the model and discuss the global qualitative behaviour of the solutions.

4. Consider the effect of regularly harvesting the population of a species for which the model equation is

$$N_{t+1} = \frac{bN_t^2}{1+N_t^2} - EN_t := f(N_t; E),$$

where E is a measure of the effort expended in obtaining the harvest, EN_t, and the parameters are such that $b > 2$ and $E > 0$.

(a) Determine the steady states and hence show that if the effort $E > E_M = (b-2)/2$, then no harvest is obtained.

(b) If $E < E_M$ show by cob-webbing $N_{t+1} = f(N_t; E)$, or otherwise, that the model is realistic only if the population, N_t, always lies between two positive values for which you should find analytic expressions (but do not solve explicitly).

(c) Demonstrate the existence of a tangent bifurcation as $E \to E_M$.

5. The interaction between two populations with densities N_1 and N_2 is modelled by

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= rN_1\left(1 - \frac{N_1}{K}\right) - aN_1N_2\left(1 - \exp[-bN_1]\right),\\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= -dN_2 + eN_2\left(1 - \exp[-bN_1]\right),\end{aligned}$$

where a, b, d, e, r and K are positive constants.

(a) What type of interaction exists between N_1 and N_2? What do the various terms imply ecologically?

(b) Non-dimensionalise the system by writing

$$u = \frac{N_1}{K}, \qquad v = \frac{aN_2}{r}, \qquad \tau = rt, \qquad \alpha = \frac{e}{r}, \qquad \delta = \frac{d}{r}, \qquad \beta = bK.$$

(c) Determine the non-negative equilibria and note any parameter restrictions.

(d) Discuss the linear stability of the equilibria.

(e) Show that a non-zero N_2 population can exist if $\beta > \beta_c = -\ln(1 - \delta/\alpha)$.

(f) Briefly describe the bifurcation behaviour as β increases with $0 < \delta/\alpha < 1$.

Optional

Consider a lake with some fish attractive to people who like to fish (from here on denoted "fishers"). We wish to model the fish-fishers interactions under the following assumptions:

6. • the fish population grows logistically in the absence of fishing;

- the presence of fishers depresses the fish growth rate at a rate jointly proportional to the size of the fish and fishers populations;
- fishing crew are attracted to the lake at a rate directly proportional to the number of fish in the lake;
- fishers are discouraged from the lake at a rate directly proportional to the number of fishers already there.

(a) Write down a mathematical model for this situation, clearly defining your terms.

(b) Show that a non-dimensionalised version of the model is

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= ru(1-u) - uv,\\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \beta u - v,\end{aligned}$$

where u and v represent the non-dimensionalised fish and fishers populations, respectively.

(c) Calculate the steady states of the system and determine their stability.

(d) Draw the phase plane, including the nullclines and phase trajectories.

(e) What would be the effect of adding fish to the lake at a constant rate?

7. (a) What kind of interactive behaviour between two populations, N_1 and N_2, is suggested by the model

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= r_1 N_1\left(1 - \frac{N_1}{K_1 + b_{12}N_2}\right),\\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= r_2 N_2\left(1 - \frac{N_2}{K_2 + b_{21}N_1}\right),\end{aligned}$$

where r_1, r_2, K_1, K_2, b_{12} and b_{21} are positive constants?

(b) Show that, with appropriate non-dimensionalisation, this model takes the form

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}\tau} &= u_1\left(1 - \frac{u_1}{1 + \alpha_{12}u_2}\right),\\ \frac{\mathrm{d}u_2}{\mathrm{d}\tau} &= \rho u_2\left(1 - \frac{u_2}{1 + \alpha_{21}u_1}\right),\end{aligned}$$

where

$$u_1 = \frac{N_1}{K_1}, \qquad u_2 = \frac{N_2}{K_2},$$

τ is non-dimensionalised time and α_{12}, α_{21} and ρ are positive parameters.

(c) Determine the steady states and their linear stability, taking care to list any restrictions on parameters.

(d) By drawing the nullclines and sketching phase trajectories, briefly discuss the behaviour of the model for the cases $\alpha_{12}\alpha_{21} < 1$ and $\alpha_{12}\alpha_{21} > 1$.