Graph Theory Part A **Problem Sheet 1** **Marc Lackenby**

1. Show that any connected graph G has a vertex v such that $G - v$ is connected.
[*You may wish to consider a spanning tree for* G.]

2. Show that a graph is minimally connected if and only if it is maximally acyclic.
[*'minimally P' means 'P holds but P doesn't hold if we delete any edge'; 'maximally P' means 'P holds but P doesn't hold if we add any new edge in the same set of vertices'*]

3. Let G be a connected graph. Show that any two paths of maximum length intersect.

4. (i) Show that in any tree T, there is a path P, such that either $T = P$ or $T \setminus P$ is a tree with fewer leaves.
(ii) Show that any tree on $n \geq 2$ vertices contains a path with k vertices or has at least n/k leaves.
Can you improve this statement?

5. Let $d_1, \ldots, d_n$ be positive integers. Show that there is a tree on n vertices with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

6. Let G be a connected graph and suppose each edge e has a positive cost $c(e)$. Show that if the costs of the edges are all distinct, then G has a unique minimum cost spanning tree.

7. Let G be a connected graph. Show that G has an Euler trail (i.e. a walk using each edge exactly once) if and only if there are at most two vertices with odd degree.

8. What is the maximum number of edges in a graph on n vertices with no triangle (ie no cycle of length 3)?
[*This question doesn't really use any of the theory of the course. But it is a classic problem with many different possible solutions. Trying to solve it will help you become more familiar with the proof techniques in graph theory.*]