

Calculus of Variations - Problem Sheet 2

Trinity Term 2021

1. It is required to find an extremal of the functional

$$\int_a^b F(x, y(x), y'(x), y''(x)) dx$$

among all smooth functions $y(x)$ which satisfy the boundary conditions

$$y(a) = y(b) = 0.$$

Show that such an extremal must be a solution of the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) = 0$$

and must satisfy the natural boundary conditions

$$\frac{\partial F}{\partial y''} = 0 \quad \text{at} \quad x = a \quad \text{and} \quad x = b.$$

2. An elastic beam has vertical displacement $y(x)$, $x \in [0, l]$. (The x -axis is horizontal and the y -axis is vertical and directed upwards.) The ends of the beam are supported, that is, $y(0) = y(l) = 0$, and the displacement minimizes the energy

$$\int_0^l \left\{ \frac{1}{2} D [y''(x)]^2 + \rho g y(x) \right\} dx,$$

where D, ρ and g are positive constants. Write down the differential equation and the boundary conditions that $y(x)$ must satisfy and show that

$$y(x) = -\frac{\rho g}{24D} x(l-x)(l^2 + x(l-x)).$$

3. Find an extremal corresponding to

$$\int_{-1}^1 y dx$$

when subject to $y(-1) = y(1) = 0$ and

$$\int_{-1}^1 (y^2 + y'^2) dx = 1$$

- 4 (a) Suppose that $F : \mathbb{R}^7 \rightarrow \mathbb{R}$ is a C^2 -function and that the C^2 -function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ gives a stationary value to the integral

$$\int \int \int_{\mathcal{V}} F(x, y, z, u, u_x, u_y, u_z) dx dy dz,$$

and satisfies $u = f$ on the smooth simple closed surface $\partial\mathcal{V}$ which bounds the open set \mathcal{V} in \mathbb{R}^3 . Show that u satisfies the Euler equation

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial u_z} \right) = \frac{\partial F}{\partial u}$$

(b) Let $\mathcal{V} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$. Find an extremal $u = u(x, y, z)$ for the problem of minimizing the integral

$$\int \int \int_{\mathcal{V}} (u_x^2 + u_y^2 + u_z^2) dx dy dz$$

when subject to the constraints

$$\int \int \int_{\mathcal{V}} u \, dx dy dz = 4\pi$$

and $u = 1$ on the boundary of \mathcal{V}

5. Let p be a positive real-valued function differentiable on the bounded interval $[a, b]$ and let q and r be positive real-valued continuous functions on $[a, b]$. Show that the extremals of

$$J(y) = \int_a^b (py'^2 + qy^2) dx$$

subject to the constraint

$$\int_a^b ry^2 dx = 1$$

must satisfy

$$(py')' + (-q + \lambda r)y = 0 \quad (A)$$

with $py' = 0$ at $x = a$ and $x = b$.

Show that if y_1 and y_2 are solutions to (A) for $\lambda = \lambda_1, \lambda_2$ respectively, where $\lambda_1 \neq \lambda_2$, then

$$\int_a^b ry_1 y_2 dx = 0. \quad (B)$$

Find the extremals of $\int_0^\pi y'^2 dx$ subject to $\int_0^\pi y^2 dx = 1$ and the corresponding values of λ . Verify that these extremals satisfy (B).