PART A SHORT OPTION: INTRODUCTION TO MANIFOLDS
TRINITY TERM 2021

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This course introduces basic results in the theory of real analysis in several variables and applies them to describe smooth submanifolds of $\mathbb{R}^n$.

1. Course Details

1.1. **Notes.** Online notes are available on the Maths Institute website:

https://courses.maths.ox.ac.uk/node/50777/materials

The lectures videos lectures are based on the online notes. The website also contains the notes for the course from 2020.

1.2. **Problem sets.** There are two problem sets, both on the course website. The first problem set is based on the material up to the statement of the Inverse Function Theorem, *i.e.* to the end of Section 4 of the online notes. The second problem set is then based on the remainder of the material in the course.

1.3. **Lecture videos.** The lecture videos cover the following material:

- Video ”0”: This reviews material from the metric spaces part of A2: Metric spaces and complex analysis. It corresponds to the material in the online notes in Section 3, up to Definition 3.8.
- Video ”0.5”: The completes the discussion of the material in Section 3.
- Video 1: This overviews the goals of the course.
- Video 2: This video introduces the definition of the derivative for functions on an open subset of a normed vector space. It corresponds to the material in Section 4 of the online notes up to Lemma 4.5.
- Video 3: This video defines directional derivatives and partial derivatives, and explains how they can be used to compute total derivative. Along with video 4 it will cover the material up to the end of Section 4.2 of the online notes.
- Video 4: This video completes the discussion of the material in Section 4.2 of the online lecture notes, proving the existence of all partial derivatives near a point along with continuity at that point is a sufficient condition for the existence of the total derivative.
- Video 5: This discusses the Chain Rule, following Section 4.3 of the online notes.
- Video 6: This video explains the notion of the gradient vector field associated to a differentiable function on an inner product space, as in Section 4.4 of the online lecture notes. In the course of discussing some simple properties of it, the video also introduces the notions of *level sets* and tangent vectors which will play an important role in the second half of the course.
- Video 7: The discusses ways in which we can extend the Mean Value Theorem for functions of a single real variable to the higher-dimensional setting, and follows Section 4.5 of the online notes.
- Video 8: The reviews the single-variable inverse function theorems from Prelims, and explains the statement of the Inverse Function Theorem for functions of several variables.
- Video 9: This video outlines the ideas in the proof of the Inverse Function Theorem, and discusses why the hypotheses of the theorem are necessary. It also introduces the definition of a diffeomorphism and of a system of local coordinates at a point.
- Video 10: This discusses the problem of solving “non-linear systems of equations”. The Implicit Function Theorem gives an answer to this question, at least locally, and a simple example explains why the hypotheses of the theorem are necessary.

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- Video 11: This video explains how to deduce the Implicit Function Theorem from the Inverse Function Theorem and how the proof in fact shows somewhat more: If Df_a has maximal rank $n-k$, then locally one can choose differentiable coordinates $(t_1, \ldots, t_n)$ in which f takes the form $f(t_1, \ldots, t_n) = (t_{n-k+1}, \ldots, t_n)$.
- Video 12: This video defines the notion of a k-submanifold M of a finite-dimensional normed vector space V. The definition requires M to be locally a level set of a C^1-function f taking values in $\mathbb{R}^{n-k}$, whose derivative Df has maximal rank (that is, rank $n-k$). The Implicit Function Theorem then shows a submanifold is also locally a graph. The tangent space T_aX and normal space $T_aX^{\perp}$ of a subset X of V at a point $a \in X$ is then defined and the basic lemma establishing its compatibility with C^1-maps is proved.
- Video 13: Using the compatibility lemma of the previous video, this video establishes some basic properties of tangent spaces, and computes some examples, and calculates the tangent space at a point of a submanifold in terms of the derivative of a function which locally defines it.
- Video 14: This video shows the graph of a C^1-function is always a submanifold, and that one could instead define a submanifold as a set locally given by the graph of a C^1-function. (It also sketches an alternative definition of a submanifold in terms of parameterizations – this is slightly more subtle than was clear from what I said in the video, so the online notes now give precise details – the remark is non-examinable however, so you can safely just ignore it!) It also describes the normal space at a point of a submanifold in the same manner.
- Video 15: Using normal spaces and tangent spaces, the video obtains a necessary condition for a point to be a local extremum of C^1-function given constraints.
- Video 16: The video gives some examples of using the Implicit Function Theorem to study level-sets, and examples of calculating tangent spaces.
- Video 17: This video gives a couple of examples of how the result on Lagrange multipliers can be applied.
- Video 18: Discusses higher derivatives, focusing on the second derivative. This is non-examinable (but given the optimisation topic, it is natural to at least consider what the second derivative is).
- Video 19: This video discusses in detail the 2016 past paper question, giving different approaches to the examiner's solution from that year (which is rather terse and thus not really a "model solution" but rather a guide for examiners for a marking scheme). It gives more details than woud be required of you in an examination and gives in some parts different approaches to the one taken in the examiner's solution.
- Video 20: (*to follow*): I will try to discuss an past paper question which focuses on the more geometric topics in the latter half of the course.