Part A: Introduction to Manifolds
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Problem Sheet 1

1. Let U, V and W be normed vector spaces, and let $\mathcal{B}(V,W)$ denote the space of bounded linear maps from V to W.

(*i*) Show that $\|.\|_\infty$ is a norm on $\mathcal{B}(V,W)$ where $\|\alpha\|_\infty := \sup_{v:\|v\|\leq 1} \|\alpha(v)\|$, and that, if $\alpha_1 \in \mathcal{B}(U,V)$, $\alpha_2 \in \mathcal{B}(V,W)$, so that $\alpha_2 \circ \alpha_1 \in \mathcal{B}(U,W)$, then $\|\alpha_2 \circ \alpha_1\|_\infty \leq \|\alpha_2\|_\infty.\|\alpha_1\|_\infty$.

(*ii*) Suppose now that V and W are finite dimensional. A map $\beta\colon V \times W \to \mathbb{R}$ is said to be *bilinear* if it is linear in each factor[1]. Viewing β as a function on $V \oplus W$ equipped with the norm $\|(v,w)\| = \|v\|+\|w\|$ $(\forall v \in V, w \in W)$, show, directly from the definition, that it is differentiable and calculate its derivative. What can you say about bilinear maps $\beta\colon V \times W \to U$ where U is an arbitrary normed vector space?

2. (i) Suppose that U is an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. Show directly from the defintion that if f is differentiable at $a \in U$, then the derivative Df_a is unique.

(ii) Let $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^4 < x_2 < x_1^2\}$ and let χ be the indicator function of Ω, so that $\chi(x_1,x_2) = 1$ if $(x_1,x_2) \in \Omega$ and is 0 otherwise. Show that all the directional derivatives of χ exist at $0_2 = (0,0)$. (*Notice that, despite this, χ is not continuous at 0_2.*)

(iii) Let $f(x_1,x_2) = \frac{x_1 x_2^2}{x_1^2+x_2^4}$ if $(x_1,x_2) \neq (0,0)$, and let $f(0,0) = 0$. Show that all the directional derivatives of f exist at 0_2 but that f is not differentiable at 0_2.

3. (i) Let P_n the space of polynomial functions $p\colon \mathbb{R}^n \to \mathbb{R}$, that is, $P_n = \mathbb{R}[x_1,\ldots,x_n]$. Show that $P_n \subseteq \mathcal{C}^1(\mathbb{R}^n,\mathbb{R})$.

(ii) Let $\det\colon \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ be the determinant function. Using part (*i*), show that det is continuously differentiable. Comparing the directional derivative of det at I_n in the direction X with the characteristic polynomial of $-X$, or otherwise, calculate $D\mathrm{det}_{I_n}$.

(iii) If $A \in \mathrm{GL}_n(\mathbb{R})$ is invertible, calculate the derivative $D\mathrm{det}_A$ of det at A, using part (*ii*) and the multiplicativity of the determinant.

4. Suppose U is an open subset of $\mathbb{R}$ and $f\colon U \to \mathbb{R}$ if differentiable on U. Show that f is continuously differentiable if and only if the function $s\colon U \times U \to \mathbb{R}$

$$s(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y}, & x \neq y, \\ f'(x), & x = y. \end{cases}$$

is continuous.

5. Let $L = \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, a normed vector space with the operator norm as in Q.1. Suppose that $I = (-r,r)$ is an open interval, and $f\colon I \to \mathbb{R}$ is given by a power series $\sum_{n\geq 0} a_n.t^n$ with radius of convergence $R > r$, so that $\sum_{n\geq 0} a_n x^n$ converges absolutely and uniformly on $(-r,r)$.

Now consider $B = B(0_L,r)$, and let $H \in B$. Let $s_n\colon L \to L$ be given by $s_n(H) = \sum_{k=0}^n a_k H^k$.

(*i*) Show that the function s_n is continuously differentiable.

[*Hint: Using the standard basis we can identify L with $Mat_n(\mathbb{R})$. What can you say about the entries of the matrix associated to $s_n(H)$?*]

(*ii*) Show that the sequence of functions $(s_n)_{n\geq 0}$ converges uniformly on $B(0_L,r)$ as $n \to \infty$. We write $s(H)$ for its limit.

(*iii*) Deduce that $s\colon B(0_L,r) \to \mathbb{R}^n$ is differentiable at $H = 0$.

6. Let $\iota\colon : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ denote the inversion map.

[1] That is $\beta(v_1+\lambda v_2,w) = \beta(v_1,w)+\lambda\beta(v_2,w)$ and similarly $\beta(v,w_1+\lambda w_2) = \beta(v,w_1)+\lambda\beta(v,w_2)$ for all $v,v_1,v_2 \in V$, $w,w_1,w_2 \in W$ and $\lambda \in \mathbb{R}$.

- (i) By considering $f(t) = 1/(1-t)$, show that if $H \in \mathrm{Mat}_n(\mathbb{R}^n)$ has $\|H\| < 1$, then $I_n - H \in \mathrm{GL}_n(\mathbb{R})$.
- (ii) Deduce that $\mathrm{GL}_n(\mathbb{R})$ is open.
- (iii) Using Q.5 or otherwise, show that ι is differentiable at I_n with derivative $D\iota_{I_n}(H) = -H$, and deduce that ι is differentiable at every $A \in \mathrm{GL}_n(\mathbb{R})$, with $D\iota_A(H) = -A^{-1}HA^{-1}$.