Part A: Introduction to Manifolds
Mathematical Institute, University of Oxford

Problem Sheet 2

1. (i) Show that there exists a real-valued $\mathcal{C}^1$-function g defined on a neighbourhood of the origin in $\mathbb{R}$ such that

$$g(x) = (g(x))^3 + 2e^{g(x)}\sin(x).$$

(ii) Show that the equations

$$e^x + e^{2y} + e^{3u} + e^{4v} = 4$$
$$e^x + e^y + e^u + e^v = 4.$$

can be solved (implicitly) for u, v in terms of x, y near the origin.

2. Let $G_n = \mathrm{GL}_n(\mathbb{R})^+$ be the open subset of $\mathrm{Mat}_n(\mathbb{R})$ consisting of those $n \times n$ invertible matrices over $\mathbb{R}$ with positive determinant. Show, directly from the definitions, that $f(A) = A^2$ is continuously differentiable on G_n. Deduce that there is some $r > 0$ for which there is a continuously differentiable function $g\colon B(I_n, r)$ such that $g(A)^2 = A$. Does such a function exist on all of G_n?

3. Let $S = \{X \in \mathrm{Mat}_2(\mathbb{R}) : \mathrm{tr}(X) = 0\}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on S by conjugation, that is $g \cdot X = gXg^{-1}$. Describe the orbits of this action determining when they are submanifolds of S and what their dimension is.
[*Hint*: consider the characteristic polynomial of X. You may assume that $\mathrm{SL}_2(\mathbb{R})$ is connected.]

4. By considering the function $f(x) = x + 2x^2\sin(1/x)$ (extended by continuity to $x = 0$), show that the hypothesis that the derivative $f'(x)$ is continuous cannot be removed.
Optional: If f strongly differentiable at $x = 0$? That is, is it true that $f(x) - f(y) = \alpha.(x-y) + o(|x-y|)$ for some $\alpha \in \mathbb{R}$?

5. Deduce the Inverse Function Theorem from the Implicit Function Theorem. (*Hint: Consider the graph of the function in the statement of the Inverse Function Theorem.*)

6. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be given by $g(x_1, \ldots, x_n) = x_1.x_2.\ldots.x_n$. Find the maximum value of g on the set $S = \{x \in \mathbb{R}^n : x_i > 0, \forall i \in \{1, \ldots, n\}, \quad \sum_{i=1}^n x_i = 1\}$.

Deduce the *arithmetic mean-geometric mean inequality*, that is, show that for positive real numbers $x_1, \ldots, x_n$ the geometric mean $(x_1.\ldots.x_n)^{1/n}$ is always less than or equal to the arithmetic mean $n^{-1}\sum_{i=1}^n x_i$.