PROJECTIVE GEOMETRY – TRINITY 2021 – SHEET 2

Duality. Symmetric Bilinear Forms. Conics. Applications to Diophantine Equations.
(Exercises on recorded lectures 5–9)

1. Write down the dual of Pappus' Theorem.

2. Let P_0, P_1, P_2, P_3 be four distinct points in a projective plane $\mathbb{P}(V)$. Show that P_0, P_1, P_2, P_3 are in general position if and only if the lines P_0P_1, P_1P_2, P_2P_3, P_3P_0 are in general position in $\mathbb{P}(V^*)$.

3. Use general position arguments to show that given five points in the projective plane, such that no three are collinear, there is a unique conic through these five points.

4. Let C, D be conics in a projective plane $\mathbb{P}(V)$, where V is a 3-dimensional real vector space, and suppose that $C \cap D = \{p_1, p_2, p_3, p_4\}$, where $p_1, \ldots, p_4$ are distinct points in $\mathbb{P}(V)$.

(a) Show that $p_1, \ldots, p_4$ are in general position. Prove that there exist homogeneous coordinates $[x_0 \colon x_1 \colon x_2]$ on $\mathbb{P}(V)$ for which

$$p_1 = [1\colon 1\colon 1], \qquad p_2 = [1\colon -1\colon 1], \qquad p_3 = [1\colon 1\colon -1], \qquad p_4 = [1\colon -1\colon -1].$$

(b) Show that any conic through $p_1, \ldots, p_4$ has equation

$$\lambda x_0^2 + \mu x_1^2 + \nu x_2^2 = 0$$

where $\lambda + \mu + \nu = 0$.

(c) Find four projective transformations τ of $\mathbb{P}(V)$ that form a group, and for which $\tau(C) = C$ and $\tau(D) = D$.

5. Let $F(x_0, x_1, x_2)$ be a homogeneous polynomial of degree n. Let $\mathcal{C}$ be the set of points $[a_0, a_1, a_2]$ in $\mathbb{RP}^2$ such that $F(a_0, a_1, a_2) = 0$. Let $\mathbf{a}$ be a point on $\mathcal{C}$. Provided that $\nabla F(\mathbf{a}) \neq \mathbf{0}$, the *tangent line* to $\mathcal{C}$ at $\mathbf{a} = [a_0, a_1, a_2]$ is the line

$$x_0 \frac{\partial F}{\partial x_0}(\mathbf{a}) + x_1 \frac{\partial F}{\partial x_1}(\mathbf{a}) + x_2 \frac{\partial F}{\partial x_2}(\mathbf{a}) = 0$$

in $\mathbb{RP}^2$ and $\mathbf{a}$ is said to be *singular* if $\nabla F(\mathbf{a}) = \mathbf{0}$.

(i) Show that $\mathbf{a}$ lies on the tangent line to $\mathbf{a}$.

(ii) Given a 3×3 symmetric real matrix B its associated *conic* is the set of solutions to the equation $\mathbf{x}^T B \mathbf{x} = 0$ where $\mathbf{x} = [x_0 \colon x_1 \colon x_2]$ and the conic is said to be *singular* if B is singular. Show that a conic is singular if and only if it has a singular point.

(iii) Sketch the curves $y^2 = x^3$ and $y^2 = x^2(x+1)$ in $\mathbb{R}^2$. What singular points do these curves have? Show that $y = x^3$ has a singular point at infinity.

6. Find all rational numbers x, y such that $x^2 + y^2 - xy = 1$.

7. Let V be a 3-dimensional real vector space and suppose that L_0, L_1, L_2, L_3 are four lines in the projective plane $\mathbb{P}(V)$ all intersecting in a common point x. Explain why

(i) if L is a line in $\mathbb{P}(V)$ that does not pass though x, but intersects L_i in a point x_i (so x_0, x_1, x_2, x_3 are four distinct collinear points), then the cross-ratio $(x_0x_1 : x_2x_3)$ is independent of the choice of L;

(ii) the cross-ratio defined in (i) equals the cross-ratio $(L_0L_1 : L_2L_3)$ formed by regarding L_0, L_1, L_2, L_3 as collinear points of the dual projective plane $\mathbb{P}(V^*)$.