# Projective Geometry

## Lecture 2: Basic definitions

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## The basic definition

We work over a general field $\mathbb{F}$. In examples, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$... We set

$$\mathbb{F}^* = \mathbb{F} \setminus \{0\}.$$

Finite fields are also of interest: $\mathbb{F} = \mathbb{F}_p$ for $p$ prime, or $\mathbb{F} = \mathbb{F}_q$ for $q$ a power of a prime.

$V$: finite-dimensional vector space over $\mathbb{F}$.

Recall: choice of basis in $V$ gives isomorphism $V \cong \mathbb{F}^{\dim V}$.

**Definition Projective space $\mathbb{P}V = \mathbb{P}(V)$ associated to $V$** is the set of one-dimensional vector subspaces of $V$.

Alternatively, more geometrically: $\mathbb{P}(V)$ is the set of lines through the origin in $V$.

## Description using an equivalence relation

A one-dimensional subspace of $V$ is of the form $\langle v \rangle$ for $v \in V \setminus \{0\}$.

Two vectors $v, w \in V \setminus \{0\}$ span the same subspace if and only if there is $\lambda \in \mathbb{F}^*$ such that

$$w = \lambda v.$$

So we can also write

$$\mathbb{P}(V) = V \setminus \{0\}/\sim,$$

where $\sim$ is the equivalence relation

$$v \sim w \iff \text{there exists } \lambda \in \mathbb{F}^* \text{ such that } w = \lambda v$$

We will write $p = [v] \in \mathbb{P}(V)$ for the equivalence class of $v \in V \setminus \{0\}$.
This is the line defined by $v$ as an point $p \in \mathbb{P}(V)$.

## Dimension of projective space

By convention,

$$\dim \mathbb{P}(V) = \dim V - 1.$$

Indeed, if $\dim V = 1$, then it is itself a line, so $\mathbb{P}(V)$ is a point of dimension 0.

A little later, we will see that the convention makes sense even when $V = \{0\}$ of dimension 0, and $\mathbb{P}(V) = \emptyset$ of dimension $-1$.

Assume that $\dim V = 2$, so $V \cong \mathbb{F}^2$.
Lines in $V$ through the origin are parameterised by their slope.

$$L = \{y = mx\} \subset \mathbb{F}^2 \text{ of slope } m \in \mathbb{F}$$

or

$$\text{the vertical line } L = \{x = 0\} \text{ of slope } \infty.$$

Hence the slope takes values in the set $\mathbb{F} \cup \{\infty\}$.
So it makes sense that $\mathbb{P}(V)$ should be of dimension 1.

# Projective linear subspaces

**Definition** A **projective linear subspace** of $\mathbb{P}(V)$ is the set $\mathbb{P}(U)$ for a linear subspace $U \subset V$.

In other words, a projective linear subspace is **the set of lines contained in a linear subspace**.

$$\dim \mathbb{P}(U) = 1 \quad \Leftrightarrow \quad \dim U = 2 \quad \Leftrightarrow \quad \mathbb{P}(U) \text{ is a projective line in } \mathbb{P}(V)$$

$$\dim \mathbb{P}(U) = 2 \quad \Leftrightarrow \quad \dim U = 3 \quad \Leftrightarrow \quad \mathbb{P}(U) \text{ is a projective plane in } \mathbb{P}(V)$$

$$\dim \mathbb{P}(U) = \dim \mathbb{P}(V) - 1 \quad \Leftrightarrow \quad \dim U = \dim V - 1 \quad \Leftrightarrow \quad \mathbb{P}(U) \text{ is a hyperplane in } \mathbb{P}(V)$$

$$\dim \mathbb{P}(U) = 0 \quad \Leftrightarrow \quad \dim U = 1 \quad \Leftrightarrow \quad \mathbb{P}(U) \text{ is a point in } \mathbb{P}(V)$$

# Line through two distinct points

Here is a basic statement. Henceforth we will assume that $\dim \mathbb{P}(V) \geq 1$.

**Lemma** Through every two distinct points $p_1, p_2 \in \mathbb{P}(V)$, there passes a unique (projective) line.

**Proof** Let $p_i = [v_i]$, then as $p_1$ and $p_2$ are distinct, $v_1$ and $v_2$ are linearly independent.
Therefore the unique projective line containing $p_1, p_2$ is

$$L = \mathbb{P}\langle v_1, v_2 \rangle.$$

□

**Note** that this is also true in the normal (affine) geometry of $\mathbb{F}^n$.

# In a projective plane, two distinct lines meet in a point

Here is another basic statement. Assume now that $\dim \mathbb{P}(V) = 2$, so we work in a projective plane.

**Lemma** Every two distinct (projective) lines $L_1, L_2 \subset \mathbb{P}(V)$ meet in a unique (projective) point.

**Proof** Let $L_i = \mathbb{P}(U_i)$ for some two-dimensional subspaces $U_i$ of the three-dimensional vector space $V$.
As the lines are different, the linear subspace $U_1 + U_2$ of $V$ is larger than each $U_i$, so we must have

$$U_1 + U_2 = V.$$

Now by the Dimension of Intersection Formula from Linear Algebra, we obtain

$$\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2) = 1.$$

So we get

$$L_1 \cap L_2 = \mathbb{P}(U_1 \cap U_2) = p \in \mathbb{P}(V)$$

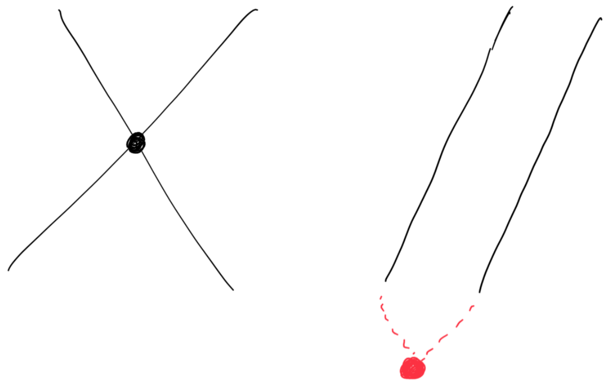
a (unique) projective point! □

## In a projective plane, two distinct lines meet in a point

Here is another basic statement. Assume now that $\dim \mathbb{P}(V) = 2$, so we work in a projective plane.

**Lemma** Every two distinct (projective) lines $L_1, L_2 \subset \mathbb{P}(V)$ meet in a unique (projective) point.

**Note** that this is **not** true in the normal (affine) geometry of $\mathbb{F}^2$, because of the existence of parallel lines.



# Intersection and projective span

Given two projective linear subspaces $\mathbb{P}(U_1)$, $\mathbb{P}(U_2)$ of $\mathbb{P}(V)$, we have their **intersection**

$$\mathbb{P}(U_1) \cap \mathbb{P}(U_1) = \mathbb{P}(U_1 \cap U_2).$$

This is the set of lines contained in both $U_1$ and $U_2$ as a subset of $\mathbb{P}(V)$.

Given two projective linear subspaces $\mathbb{P}(U_1)$, $\mathbb{P}(U_2)$ of $\mathbb{P}(V)$, define their **linear span**

$$\langle \mathbb{P}(U_1), \mathbb{P}(U_1) \rangle = \mathbb{P}(U_1 + U_2).$$

This has a geometric interpretation as the union of projective lines spanned by a point of $\mathbb{P}(U_1)$ and a point of $\mathbb{P}(U_2)$ inside $\mathbb{P}(V)$ (see Problem Sheet 1).

# The Dimension Formula

**Theorem (Projective Dimension of Intersection Formula)**
Let $L_1, L_2$ be two projective linear subspaces of a projective space $\mathbb{P}(V)$. Then we have

$$\dim(L_1 \cap L_2) = \dim(L_1) + \dim(L_2) - \dim\langle L_1, L_2\rangle.$$

Here the convention $\dim \emptyset = -1$ is in force.

**Proof** Let $L_i = \mathbb{P}(U_i)$. Then we have, by the Dimension of Intersection Formula in Linear Algebra,

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2).$$

Subtracting one from each term gives the result. □

# An example

**Example** Given two (projective) planes $P_1, P_2$ in a four-dimensional projective space $\mathbb{P}(V)$, then

1. $P_1 = P_2$, or
2. $P_1 \cap P_2 = L$ is a line, and $\dim\langle P_1, P_2\rangle = 3$, or
3. $P_1 \cap P_2 = p$ is a point, and $\langle P_1, P_2\rangle = \mathbb{P}(V)$.

This follows from the Dimension Formula, distinguishing cases.
Note: once again, no exceptions for “parallel” planes.

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