Projective Geometry

Lecture 3: Projective coordinates

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Recollections on projective space

We start with a field $\mathbb{F}$, and a finite-dimensional $\mathbb{F}$-vector space V.
Recall

$$\begin{aligned}\mathbb{P}(V) &= \{\text{one-dimensional vector subspaces of } V\} \\ &= V \setminus \{0\}/(v \sim \lambda v, \quad \lambda \in \mathbb{F}^*)\end{aligned}$$

Assume $\dim V = n+1$, $\dim \mathbb{P}(V) = n$.

If we fix a basis $\{e_0, \dots, e_n\}$ of V, then

$$V \cong \mathbb{F}^{n+1}.$$

We will denote

$$\mathbb{FP}^n = \mathbb{P}(\mathbb{F}^{n+1}).$$

Coordinates on projective space

Fixing a basis $\{e_0, \ldots, e_n\}$ of V, for $v \in V$ there is a unique decomposition

$$v = \sum_{i=0}^{n} x_i e_i.$$

Then

$$v \sim \lambda v \Longrightarrow (x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n).$$

We get

$$\mathbb{FP}^n = \mathbb{F}^{n+1} \setminus \{0\}/(v \sim \lambda v, \quad \lambda \in \mathbb{F}^*).$$

Denote the equivalence class of this vector

$$[v] = [x_0 : x_1 : \ldots : x_n].$$

Coordinates on projective space

We get **projective coordinates**

$$p = [v] = [x_0 : x_1 : \ldots : x_n] \in \mathbb{F}\mathbb{P}^n.$$

Rules on projective coordinates:

(1) a point in projective space $p \in \mathbb{F}\mathbb{P}^n$ has $\dim \mathbb{P}(V)+1$ coordinates $(x_0, \ldots, x_n)$;

(2) $x_i \in \mathbb{F}$;

(3) not all x_i are 0;

(4) $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for $\lambda \in \mathbb{F}^*$.

The projective line in coordinates

Take $n = 1$, so we are looking at $\mathbb{FP}^1 = \mathbb{P}(\mathbb{F}^2)$.
A point $p \in \mathbb{FP}^1$ has coordinates $[x_0 : x_1]$.

(1) $x_0 \neq 0$. Then

$$p = [x_0 : x_1] = [1 : x_1/x_0] = [1 : \alpha] \text{ for } \alpha \in \mathbb{F}.$$

(2) $x_0 = 0$. Then

$$p = [0 : x_1] = [0 : 1] \text{ a unique point.}$$

We get a bijection

$$\mathbb{FP}^1 \longleftrightarrow \mathbb{F} \sqcup \{*\}$$

which we often write as

$$\mathbb{FP}^1 \longleftrightarrow \mathbb{F} \sqcup \{\infty\}$$

with

$$[0 : 1] = \text{ the point at } \infty.$$

The point at infinity

We have a bijection

$$\mathbb{FP}^1 \longleftrightarrow \mathbb{F} \sqcup \{\infty\}.$$

We obtained this in the discussion of slopes of lines in $\mathbb{F}^2$ in the previous lecture!

Important: this decomposition depends on **choices**.

1. We chose a basis on our two-dimensional vector space V to get $V \cong \mathbb{F}^2$.
2. Then we chose x_0 to be our distinguished coordinate.

There is no distinguished point on $\mathbb{P}(V)$ for V a two-dimensional vector space.

Any point $p \in \mathbb{P}(V)$ could serve as the point ∞.

For if $p = [v]$, then we can choose a basis

$$(e_0, e_1) = (v, w)$$

and then $p = [0w + 1v] = [0 : 1]$.

The point at infinity in the complex projective line

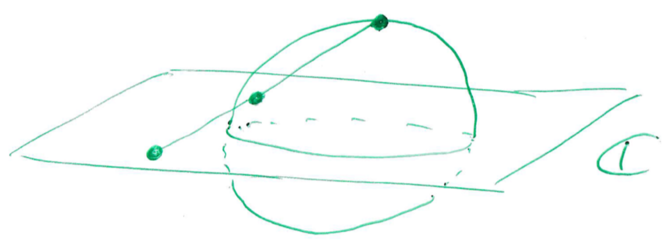
For $\mathbb{F} = \mathbb{C}$, we get a bijection

$$\mathbb{CP}^1 \longleftrightarrow \mathbb{C} \sqcup \{\infty\}.$$

This is nothing but the Riemann sphere!
Dfferent points of view on the Riemann sphere:

(1) $\mathbb{CP}^1$ is the **complex projective line**, a **one-dimensional object** over $\mathbb{C}$.

(2) $\mathbb{C} \sqcup \{\infty\}$ is a **sphere** (via stereographic projection), a **two-dimensional object** in real coordinates, so over $\mathbb{R}$.



The projective plane in coordinates

Next, take $n = 2$, so we are looking at $\mathbb{FP}^2 = \mathbb{P}(\mathbb{F}^3)$.
A point $p \in \mathbb{FP}^2$ has coordinates $[x_0 : x_1 : x_2]$.

(1) $x_0 \neq 0$. Then

$$p = [x_0 : x_1 : x_2] = [1 : x_1/x_0 : x_2/x_0] = [1 : \alpha : \beta] \text{ for } \alpha, \beta \in \mathbb{F}.$$

(2) $x_0 = 0$. Then

$$p = [0 : x_1 : x_2] \text{ a point in } \mathbb{FP}^1.$$

We get a bijection

$$\mathbb{FP}^2 \longleftrightarrow \mathbb{F}^2 \sqcup \mathbb{FP}^1$$

which we can think of as

$$\mathbb{FP}^2 \longleftrightarrow \mathbb{F}^2 \sqcup \{\text{line at } \infty\}$$

with

$$[0 : x_1 : x_2] = \text{ideal point on line at } \infty \text{ in direction } [x_1 : x_2].$$

The line at infinity in the projective plane

Once again, the decomposition

$$\mathbb{FP}^2 \longleftrightarrow \mathbb{F}^2 \sqcup \{\text{line at } \infty\}$$

depends on choices:

1. We chose a basis on our three-dimensional vector space V to get $V \cong \mathbb{F}^3$.
2. Then we chose x_0 to be our distinguished coordinate.

In the same way as before, any projective line (one-dimensional projective linear subspace) $L \subset \mathbb{FP}^2$ can serve as line at infinity, with

$$\mathbb{FP}^2 \longleftrightarrow \mathbb{F}^2 \sqcup L.$$

A decomposition of projective space

Take arbitrary n, with $\mathbb{FP}^n = \mathbb{P}(\mathbb{F}^{n+1})$.
A point $p \in \mathbb{FP}^n$ has coordinates $[x_0 : x_1 : \ldots : x_n]$.

(1) $x_0 \neq 0$. Then

$$p = [x_0 : \ldots : x_n] = [1 : x_1/x_0 : \ldots : x_n/x_0] = [1 : \alpha_1 : \ldots : \alpha_n] \text{ for } (\alpha_i) \in \mathbb{F}^n.$$

(2) $x_0 = 0$. Then

$$p = [0 : x_1 : \ldots x_n] \text{ a point in } \mathbb{FP}^{n-1}.$$

We get a bijection, **depending on choices**

$$\mathbb{FP}^n \longleftrightarrow \mathbb{F}^n \sqcup \mathbb{FP}^{n-1}$$

which we can think of as

$$\mathbb{FP}^n \longleftrightarrow \mathbb{F}^n \sqcup \{\text{hyperplane at } \infty\}$$

with

$[0 : x_1 : \ldots : x_n]$ = ideal point on hyperplane at ∞ in direction $[x_1 : \ldots : x_n]$.

Projective linear subspaces

Recall that projective linear subspaces in $\mathbb{P}(V)$ are $\mathbb{P}(U)$ for $U \leq V \cong \mathbb{F}^{n+1}$ a linear subspace.

One way to view these is as zero-locus of linear, homogeneous equations

$$U = \left\{ v = (x_i) : \sum_{i=0}^{n} \alpha_{ji} x_i = 0,\ j = 1, \ldots, m \right\} = \ker T \leq V$$

for a linear map

$$T \colon V \to \mathbb{F}^m$$

given by the matrix $A = (\alpha_{ji})$.

We can write directly

$$\mathbb{P}(U) = \left\{ [x_i] : \sum_{i=0}^{n} \alpha_{ji} x_i = 0,\ j = 1, \ldots, m \right\} \subset \mathbb{P}(V)$$

Note that the equations in blue make sense on coordinates **up to scale**.

Lines in the plane, again

Go back to the case of the projective plane.

$$\begin{array}{rcl} \mathbb{FP}^2 & \longleftrightarrow & \mathbb{F}^2 \sqcup \mathbb{FP}^1 \\ [x_0 : x_1 : x_2] & \mapsto & [1 : x_1/x_0 : x_2/x_0] \text{ if } x_0 \neq 0 \\ [0 : x_1 : x_2] & \mapsto & [x_1 : x_2] \in \mathbb{FP}^1 \\ \\ [1 : x : y] & \leftarrow\!\shortmid & (x, y) \end{array}$$

Consider affine line

$$l = \{y = 2x + 1\} \subset \mathbb{F}^2.$$

To get the corresponding projective line in $\mathbb{FP}^2$, we substitute $x = x_1/x_0$, $y = x_2/x_0$ to get, clearing denominator,

$$L = \{x_2 = 2x_1 + x_0\} \subset \mathbb{FP}^2.$$

Then we have

$$L = l \sqcup \{[0 : 1 : 2]\}.$$

Lines in the plane, again

More generally, the “slope m” line

$$l = \{y = mx + c\} \subset \mathbb{F}^2$$

becomes the projective line

$$L = \{x_2 = mx_1 + cx_0\} \subset \mathbb{FP}^2$$

with decomposition

$$L = l \sqcup \{[0 : 1 : m]\}.$$

The “vertical” line of ”slope infinity”

$$l = \{x = d\} \subset \mathbb{F}^2$$

becomes

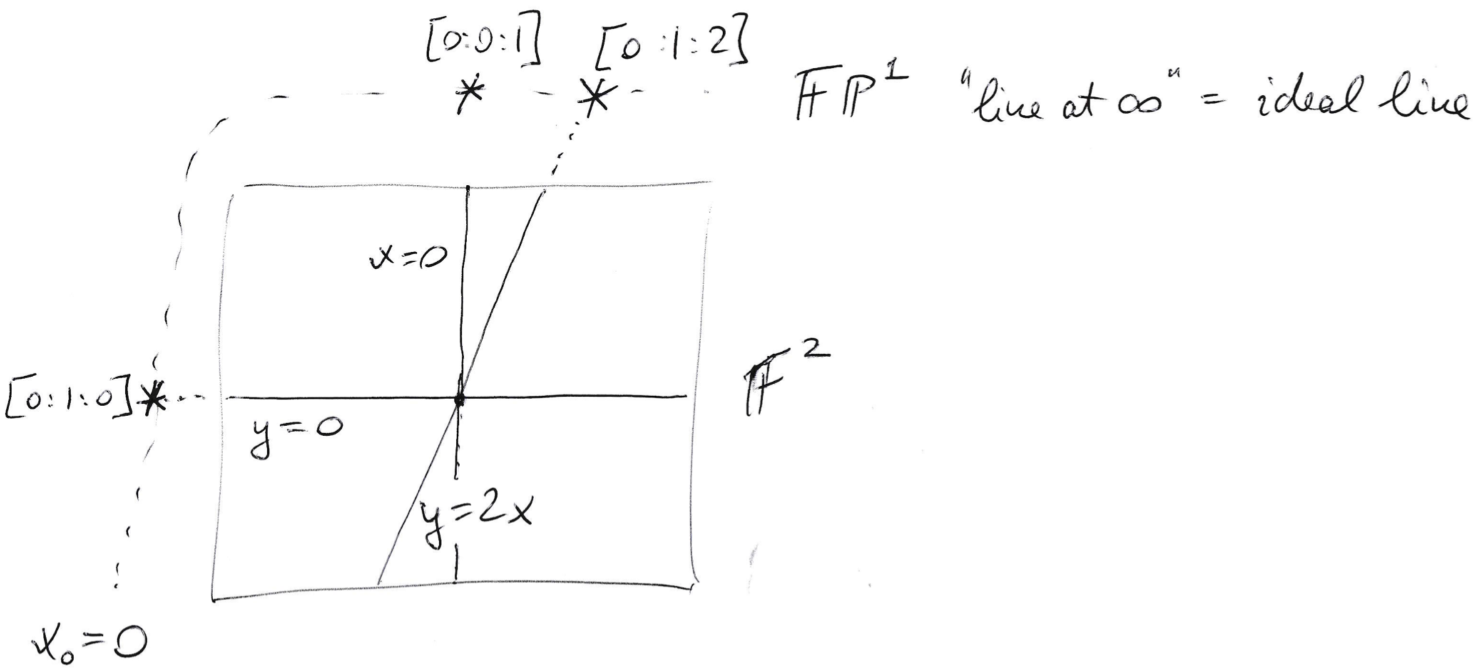
$$L = \{x_1 = dx_0\} = l \sqcup [0 : 0 : 1] \subset \mathbb{F}^2.$$

Finally there is the ”ideal line at infinity”

$$L_\infty = \{x_0 = 0\} \subset \mathbb{FP}^2$$

which cannot be obtained from a line $l \subset \mathbb{F}^2$ in this picture.

Lines in the plane, again



Real projective spaces

Finally let us look at the case $\mathbb{F} = \mathbb{R}$. Recall the n-sphere

$$S^n = \{x_0^2 + \ldots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}.$$

As we can scale real vectors to real vectors of unit length, we have

$$\begin{aligned} \mathbb{RP}^n &= \mathbb{R}^{n+1} \setminus \{0\}/(v \sim \lambda v, \quad \lambda \in \mathbb{R}^*) \\ &= S^n/(v \sim -v). \end{aligned}$$

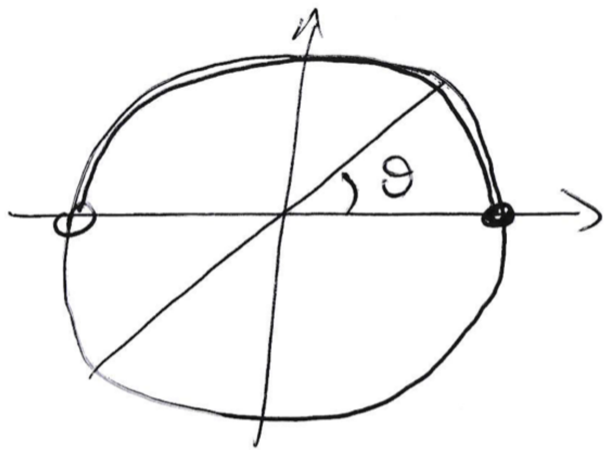
Slogan: $\mathbb{RP}^n$ is the same as S^n with its **antipodal points identified**.

Corollary As a topological space, $\mathbb{RP}^n$ is connected and compact.

Proof The topological space S^n has these properties, and $\mathbb{RP}^n$ is a surjective image of S^n under a continuous map.

Real projective line

Example 1: $\mathbb{RP}^1$ is the same as S^1 with direction (angle) θ identified with direction (angle) $\theta + \pi$.



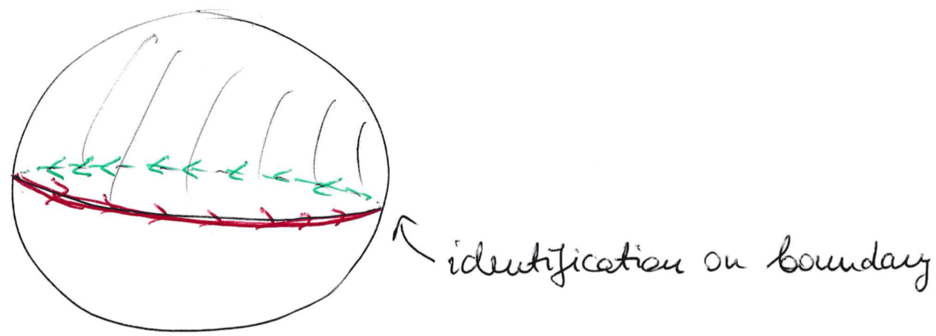
In other words, points of the circle can be parametrized by $\theta \in \mathbb{R} \mod 2\pi$, whereas directions in the plane are parametrized by $[\theta] \in \mathbb{R} \mod \pi$.

Real projective plane

Example 2: For the real projective plane $\mathbb{RP}^2$, we get

$$\begin{aligned}\mathbb{RP}^2 &= S^2/(v \sim -v)\\ &= U^2/(v \sim -v)\end{aligned}$$

where U^2 is the **upper hemisphere**, and the identification happens only **along its boundary**.



This picture will be familiar to those who took the Topology course last term.