Projective Geometry

Lecture 4: Projective transformations

Balázs Szendrői, University of Oxford, Trinity term 2021

Maps for our objects

We want to describe **maps** between our objects: **projective transformations** on **projective spaces**.

We have defined projective space $\mathbb{P}V$ in terms of an equivalence relation on a vector space as

$$\mathbb{P}V = V \setminus \{0\}/(v \sim \lambda v \colon \lambda \in \mathbb{F}^*).$$

Natural guess: consider maps of projective spaces induced by linear maps of vector spaces.
So assume $T \colon V \to W$ is a linear map, and let the rule

$$\tau : [v] \mapsto [T(v)]$$

define a map from $\mathbb{P}V$ to $\mathbb{P}W$.

Maps: potential issues

There are two potential problems.

- Q1: Is the map
$$\tau : [v] \mapsto [T(v)]$$
well-defined on $\mathbb{P}V$, i.e. on equivalence classes $[v] \in \mathbb{P}V$?

  A1: Yes!

- Q2: Is the map
$$\tau : [v] \mapsto [T(v)]$$
well-defined as a map to $\mathbb{P}W$?

  A2: not necessarily! We need T to be **injective**.

Projective transformations: the definition

Definition A **projective transformation**

$$\tau : \mathbb{P}V \to \mathbb{P}W$$

attached to an **injective** linear map $T\colon V \to W$ is the map defined by the rule

$$\tau : [v] \mapsto [T(v)].$$

Special case: the projective transformation

$$\tau : \mathbb{P}V \to \mathbb{P}V$$

attached to an **invertible** linear map $T\colon V \to V$.
Such transformations are automatically invertible themselves: τ^{-1} is the projective transformation attached to the linear map T^{-1}.

The group of projective transformations

For a fixed $\mathbb{F}$-vector space V, the projective transformations

$$\tau : \mathbb{P}V \to \mathbb{P}V$$

form a group:

1. The identity transformation is attached to Id_V.
2. These projective transformations are invertible.
3. A composite of two projective transformations is a projective transformation.
4. Composition is automatically associative.

Thus, to every projective space $\mathbb{P}V$, we have attached the group

$$\mathrm{PGL}(V) = \{\tau : \mathbb{P}V \to \mathbb{P}V \text{ a projective transformation}\}.$$

The group of projective transformations on the projective line

Example Let $V = \mathbb{F}^2$, so $\mathbb{P}V = \mathbb{FP}^1$.
The linear map $T\colon \mathbb{F}^2 \to \mathbb{F}^2$ is given by an invertible 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We get

$$\begin{array}{rcccl} \tau & : & \mathbb{FP}^1 & \to & \mathbb{FP}^1 \\ & & [x_0 : x_1] & \mapsto & [(ax_0 + bx_1)\colon (cx_0 + dx_1)]. \end{array}$$

We can write this as

$$[x_0/x_1 : 1] \mapsto [(ax_0 + bx_1)/(cx_0 + dx_1)\colon 1]$$

at least all the denominators are nonzero, or, using the coordinate $x = x_0/x_1$,

$$[x : 1] \mapsto [(ax + b)/(cx + d) : 1].$$

The group of projective transformations on the projective line

We have

$$\tau\colon [x:1] \mapsto [(ax+b)/(cx+d):1].$$

So if we write

$$\begin{array}{ccccc} \mathbb{FP}^1 & = & \{x_1 \neq 0\} & \sqcup & \{[1:0]\} \\ & & \mathbb{F} & \sqcup & \{\infty\} \end{array}$$

we simply get, on the $\mathbb{F}$ part,

$$\tau\colon x \mapsto \frac{ax+b}{cx+d}.$$

In other words, **projective transformations of a projective line** are...

...**Möbius transformations**!

The group of projective transformations on the projective plane

Example Let $V = \mathbb{F}^3$, so

$$\mathbb{P}V = \mathbb{FP}^2 = \mathbb{F}^2 \sqcup L_\infty$$

with the "finite" part given by $x_0 \neq 0$ and $L_\infty = \{x_0 = 0\}$.
Let $\tau\colon \mathbb{FP}^2 \mapsto \mathbb{FP}^2$ be given by a special matrix

$$\begin{pmatrix} 1 & 0 \\ \mathbf{b} & A \end{pmatrix}$$

with $\mathbf{b} \in \mathbb{F}^2$ and A an invertible 2×2 matrix. Then an easy calculation gives, for $\mathbf{x} = (x_1, x_2) \in \mathbb{F}^2$,

$$[1 : \mathbf{x}] \mapsto [1 : A\mathbf{x} + \mathbf{b}].$$

In other words, **projective transformations** include...

...**affine linear transformations**!

Remark It is not hard to show (exercise!) that these are all the projective transformations of $\mathbb{FP}^2$ mapping L_∞ to itself.

The structure of the group of projective transformations

We have

$$\mathbb{P}V = V \setminus \{0\}/(v \sim \lambda v \colon \lambda \in \mathbb{F}^*).$$

Proposition We also have

$$\mathrm{PGL}(V) = \mathrm{GL}(V)/(T \sim \lambda T \colon \lambda \in \mathbb{F}^*).$$

Proof It is clear that $T, \lambda T$ define the same map

$$[v] \mapsto [T(v)] = [(\lambda T)(v)]$$

on $\mathbb{P}V$.

The structure of the group of projective transformations

Proposition

$$\mathrm{PGL}(V) = \mathrm{GL}(V)/(T \sim \lambda T \colon \lambda \in \mathbb{F}^*).$$

Proof continued Conversely, suppose $T_1, T_2 \in \mathrm{GL}(V)$ define the same map on $\mathbb{P}V$, so

$$[T_1(v)] = [T_2(v)] \text{ for all } v \in V.$$

Take $v, w \in V$ linearly independent. Then there are constants $\lambda, \mu, \nu \in \mathbb{F}^*$ with

$$\begin{aligned} T_2(v) &= \lambda T_1(v) \\ T_2(w) &= \mu T_1(w) \\ T_2(v+w) &= \nu T_1(v+w). \end{aligned}$$

We get

$$0 = (\lambda - \mu) T_1(v) + (\lambda - \nu) T_1(w).$$

By linear independence of v, w and invertibility of T_1, we get $\lambda = \mu = \nu$.

Proposition

$$\mathrm{PGL}(V) = \mathrm{GL}(V)/(T \sim \lambda T \colon \lambda \in \mathbb{F}^*).$$

Proof concluded So for every vector v, we get

$$T_2(v) = \lambda T_1(v)$$

with a **fixed** constant $\lambda \in \mathbb{F}^*$. So

$$T_2 = \lambda T_1 \text{ with } \lambda \in \mathbb{F}^*.$$

□

The structure of the group of projective transformations

For group theorists Recall that the centre of the General Linear Group is

$$Z(\mathrm{GL}(V)) = \mathbb{F}^* \cdot \mathrm{Id}_V.$$

Perhaps this is more familiar as the slogan **a matrix that commutes with all other (invertible) matrices is (invertible) constant diagonal**.

We see that we can view

$$\mathrm{PGL}(V) \cong \mathrm{GL}(V)/Z(\mathrm{GL}(V))$$

as the quotient group of the General Linear Group of V by its centre.

Example For $\mathbb{F} = \mathbb{F}_p$, we get very interesting finite groups in this way.

One of these, $\mathrm{PSL}(2, \mathbb{F}_7)$ is a non-abelian finite simple group of order 168, the second smallest possible after A_5 of order 60.

The action of Möbius transformations

Recall: the group of Möbius transformations acts **triply transitively** on the set $\mathbb{C} \sqcup \infty$.

Namely, given any $z_0, z_1, z_2 \in \mathbb{C}_\infty$ and $w_0, w_1, w_2 \in \mathbb{C}_\infty$ distinct complex numbers (including infinity), there is a **unique** Möbius transformation φ with $\varphi(z_i) = w_i$.

One proof proceeds via the special case $w_0, w_1, w_2 = 0, 1, \infty$ in which case φ can be written down "by hand":

$$\varphi(z) = \frac{z_1 - z_2}{z_1 - z_0} \frac{z - z_0}{z - z_2}.$$

The same argument gives the same result for an arbitrary field $\mathbb{F}$.

Proposition The group $\mathrm{PGL}(\mathbb{F}^2)$ acts sharply triply transitively on $\mathbb{FP}^1$. That is, given any two triples $p_0, p_1, p_2 \in \mathbb{FP}^1$ and $q_0, q_1, q_2 \in \mathbb{FP}^1$ of distinct points, there exists a unique $\tau \in \mathrm{PGL}(\mathbb{F}^2)$ with

$$\tau(p_i) = q_i.$$

Points in general position

We want a higher-dimensional generalization of the condition that a triple of points $p_0, p_1, p_2 \in \mathbb{FP}^1$ should consist of **distinct** points.

Definition In an n-dimensional projective space $\mathbb{P}(V)$ for an $(n+1)$-dimensional vector space V over $\mathbb{F}$, we say that $(n+2)$ points $p_0, \ldots, p_{n+1}$ are **in general position**, if each subset of $n+1$ of these points is represented by linearly independent representative vectors.

In the language of the projective span, we can translate this linear algebraic condition into the requirement that each subset of $n+1$ of these points should have $\mathbb{P}(V)$ as their projective span.

As a subset of a linearly independent set is also linearly independent, we see that the condition is also equivalent to the following: **every $k+1$-subset of $p_0, \ldots, p_{n+1}$** should span a **k-dimensional projective subspace** for $k \leq n$.

For $n = 2$, the condition means that for three points p_0, p_1, p_2 on $\mathbb{FP}^1$, each two should span the whole line, which means that they should be distinct.

General position theorem

Theorem (General Position Theorem) Let $p_0, \ldots, p_{n+1}$, respectively $q_0, \ldots, q_{n+1}$ be two $(n+2)$-tuples of points in n-dimensional projective space $\mathbb{P}(V)$, with both $(n+2)$-tuples in general position.
Then there exists a unique projective transformation $\tau \in \mathrm{PGL}(V)$ such that

$$\tau(p_i) = q_i$$

for each i.

Proof (existence) Let $p_i = [v_i]$ for $i = 0, \ldots, n+1$.
The general position hypothesis implies that $v_0, \ldots, v_n$ form a basis for the vector space V. Then for the last vector, we have

$$v_{n+1} = \sum_{i=0}^{n} \lambda_i v_i$$

for some scalars λ_i.

General position theorem

Proof (existence continued) Now, all λ_i are nonzero, again using the general position hypothesis: if one were to be zero, then we would get a dependency relation between v_{n+1} and n of the other v_i. So we may replace v_i by $\lambda_i v_i$ and take

$$v_{n+1} = \sum_{i=0}^{n} v_i$$

as representative vector for our last point. Again using the general position hypothesis, this representation of v_{n+1} is unique.
Similarly we can take $q_i = [w_i]$ for $i = 0, \ldots, n+1$, with

$$w_{n+1} = \sum_{i=0}^{n} w_i,$$

where $w_0, \ldots, w_n$ is another basis of V.

General position theorem

Proof (existence continued) There exists an invertible linear transformation T of the $(n+1)$-dimensional space V with

$$T(v_i) = w_i$$

for $i = 0, \ldots, n$. Linearity and the formulae for v_{n+1}, w_{n+1} imply that

$$T(v_{n+1}) = w_{n+1}$$

also. We deduce that the attached projective transformation indeed maps

$$\tau(p_i) = q_i$$

for each i.

General position theorem

Proof (uniqueness)

If S is another linear transformation inducing a projective transformation with the required property, then

$$Sv_i = \mu_i w_i$$

for $i = 0, \ldots, n+1$, where μ_i are nonzero scalars. Now

$$\mu_{n+1} w_{n+1} = S v_{n+1} = \sum_{i=0}^{n} S v_i = \sum_{i=0}^{n} \mu_i w_i,$$

so $w_{n+1} = \sum_{i=0}^{n} (\mu_i / \mu_{n+1}) w_i$.

By uniqueness of this representation we see all the μ_i are equal to some constant $\mu \in \mathbb{F}^*$.

Hence $S = \mu T$ and they induce the same projective map. $\square$

General position theorem: examples

Example For $n = 2$, we recover the result that $\mathrm{PGL}(\mathbb{F}^2)$ acts sharply triply transitively on $\mathbb{FP}^1$: any two triples of distinct points are projectively equivalent by a unique projective transformation.

Example Consider $n = 3$. General position for a quadruple p_0, p_1, p_2, p_3 of points of $\mathbb{FP}^2$ means: no three points lie on a line.

So we get the result that on the projective plane, **any two proper quadrangles are projectively equivalent** by a unique projective transformation.

Recall that in Euclidean geometry, we have squares, rhombi, deltoids, rectangles, trapeziums, parallelograms, convex and concave quadrangles... all these differences **disappear** in the **geometry of the projective plane**.

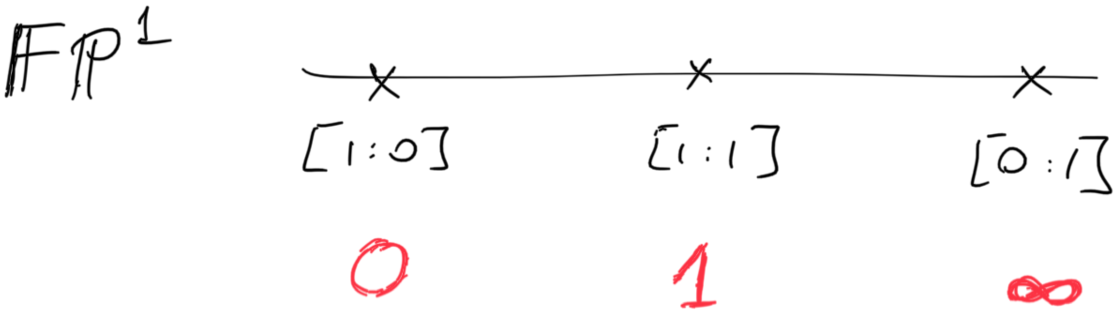
Choosing adapted coordinates

Corollary of the proof Given $p_0, \ldots, p_{n+1}$ in n-dimensional projective space $\mathbb{P}(V)$ in general position, there exists a coordinate system on $\mathbb{P}(V)$ in which the coordinates of the points are

$$\begin{aligned} p_0 &= [1 : 0 : 0 : \ldots : 0 : 0] \\ p_1 &= [0 : 1 : 0 : \ldots : 0 : 0] \\ \ldots & \\ p_n &= [0 : 0 : 0 : \ldots : 0 : 1] \\ p_{n+1} &= [1 : 1 : 1 : \ldots : 1 : 1] \end{aligned}$$

This is very useful in computational arguments.

Adapted coordinates on the projective line



Adapted coordinates on the projective plane