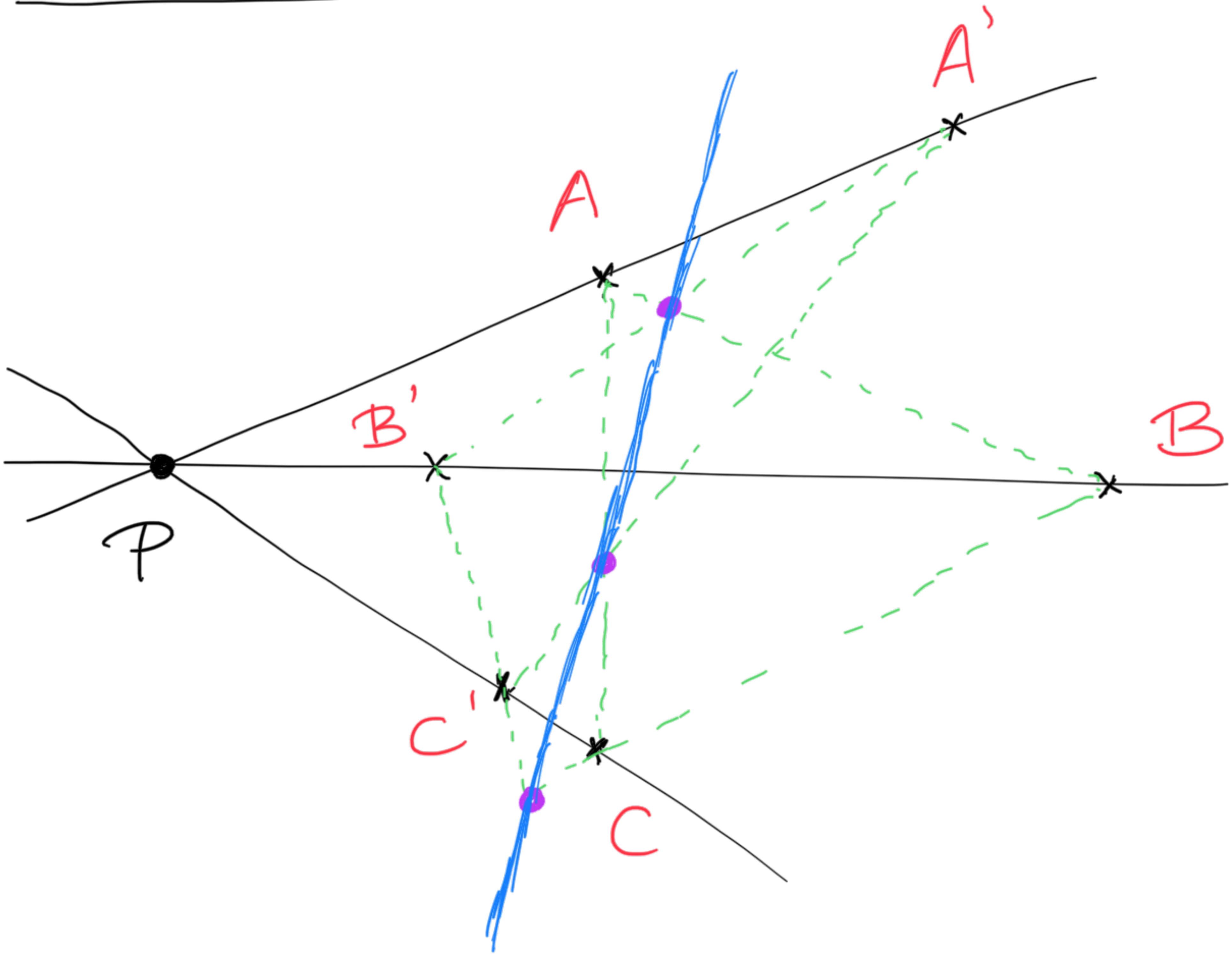
PROJECTIVE GEOMETRY LECTURE 5



__Theorem__ (Desargues) Let P, A, A', B, B', C, C' be distinct points of $\mathbb{F}\mathbb{P}^n$, such that the lines AA', BB' and CC' are all distinct, and all meet at P.

Then the points of intersection $AB \wedge A'B'$, $AC \wedge A'C'$ and $BC \wedge B'C'$ are collinear.

<u>Proof</u>: A, A', P collinear, and distinct, so their representing vectors a, a', p satisfy a relation

$$p = a + a'$$ (by adjusting coefficients)

Similarly, as B, B', P are collinear, some representing vectors b, b' of B, B' satisfy

$$p = b + b'$$ (by adj. coeff's)

Finally also

$$p = c + c'$$ (— — of vectors for C, C')

Now $\quad a + a' = b + b' \quad$ so

$$a - b = b' - a'$$

so $\quad [a-b] = [b'-a'] \in \mathbb{FP}^n$

$$\begin{array}{ccc} \cap & & \cap \\ AB & & A'B' \end{array} \quad \text{(lines)}$$

<u>Hence</u> $\quad AB \cap A'B' = [\underline{a-b}] = [b'-a']$

Similarly $\quad AC \cap A'C' = [\underline{a-c}] = [c'-a']$

$$BC \cap B'C' = [\underline{b-c}] = [c'-b']$$

Now $\quad (a-b) + (b-c) + (c-a) = 0$

hence $[a-b]$, $[b-c]$, $[c-a]$ all lie on a line as claimed! $\square$

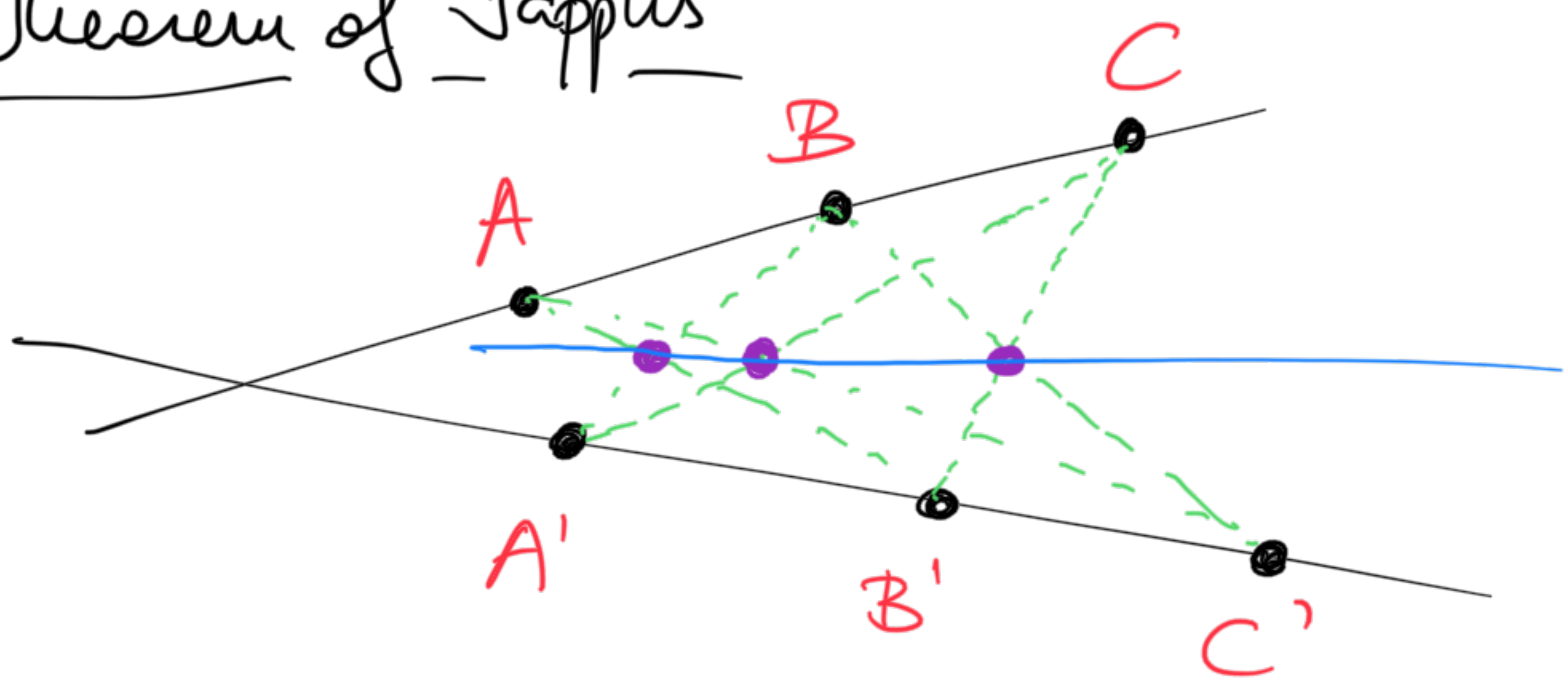
Proof 2 Step 1 $n \geq 3$, so we are not in a projective plane, and $\langle P, A, A', \dots, C, C' \rangle$ has dimension ≥ 3.
Analyse sets of points and their span - using dim. of intersection, deduce that blue line exists.

Step 2 $n = 2$, then add a third dimension,

$$\mathbb{FP}^2 \hookrightarrow \mathbb{FP}^3$$

lift configuration "into space". Use Step 1 to conclude.

Theorem of Pappus



Let A, B, C, respectively A', B', C' be collinear triples of distinct points in $\mathbb{F}\mathbb{P}^n$. Then the three points $AB' \cap A'B$, $AC' \cap A'C$ and $BC' \cap B'C$ are collinear.

__Proof__ One computational proof is on Problem Sheet - use initial simplification using the Corollary from last lecture.

__Cross-ratio__ - configurations of 4 points on $\mathbb{F}\mathbb{P}^1$.

Let P_0, P_1, P_2, P_3 be points on $\mathbb{F}\mathbb{P}^1 = \mathbb{P}(\mathbb{F}^2)$

Write $P_i = [x_i : y_i]$ for $x_0, \dots, y_3 \in \mathbb{F}$.

Assume $P_i \neq P_j$, so $\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \neq 0$ for $i \neq j$. (+)

Define the __cross-ratio__ of (P_0, P_1, P_2, P_3)

$$(P_0 P_1 : P_2 P_3) = \frac{x_0 y_2 - x_2 y_0}{x_0 y_3 - x_3 y_0} \, \frac{x_1 y_3 - x_3 y_1}{x_1 y_2 - x_2 y_1} \in \mathbb{F}^*$$

- denominators do not vanish because of (+)
- numerators — . . . .
- well-defined if we rescale $[x_i : y_i]$

<u>Key property</u>:

<u>Lemma</u> $(P_0 P_1 : P_2 P_3)$ is invariant under $PGL(\mathbb{F}^2)$, so under projective transformations.

<u>Proof</u> $\tau \in PGL(\mathbb{F}^2)$ comes from

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{F}^2)$$

Also $\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$

using matrix multiplication.

So $\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \longmapsto \underbrace{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\neq 0} \cdot \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$

Hence $(P_0P_1 : P_2P_3)$ remains unchanged. $\square$

<u>Interpretation</u>: $(P_0P_1 : P_2P_3)$ is an <u>invariant</u>

<u>quantity</u> of four points on $\mathbb{F}\mathbb{P}^1$

under projective transformations

Note: any two quadruples of points in general position on $\mathbb{FP}^2$ are equivalent under $PGL(\mathbb{F}^3)$

So quadruples of points in the plane (in general position) have no invariant quantity!

No length function in proj. geom, but along a line, four points give rise to this "ratio" - cross-ratio!

Theorem Two quadruples of distinct points on $\mathbb{FP}^1$ are projectively equivalent if and only if their cross-ratios are equal.

ie: P_0, P_1, P_2, P_3 are different pts on $\mathbb{FP}^1$
$Q_0, Q_1, \ldots, Q_3$ - - - - - - - - - then

$$\exists\ \tau \in PGL(\mathbb{F}^2) \text{ with } \tau(P_i) = Q_i \iff (P_0 P_1 : P_2 P_3) = (Q_0 Q_1 : Q_2 Q_3)$$

Proof $\Longrightarrow$ is the Lemma above.
$\Longleftarrow$ Use Corollary from last time! (P_0, P_1, P_2) are three distinct points on $\mathbb{FP}^1$, so in general position, so there is a coordinate system in which

$$P_0 = [1:0], \quad P_1 = [0:1], \; P_2 = [1:1] \quad , P_3 = [x_3 : y_3]$$

Now $x_3 \neq 0$, $y_3 \neq 0$, also $x_3 : y_3 \neq 1:1$ as $P_2 \neq P_3$.

$$(P_0 P_1 : P_2 P_3) = \frac{x_3}{y_3} = \lambda$$

In a suitable coordinate system,

$P_0 = [1:0]$, $P_1 = [0:1]$, $P_2 = [1:1]$, $P_3 = [\underline{\lambda} : 1]$

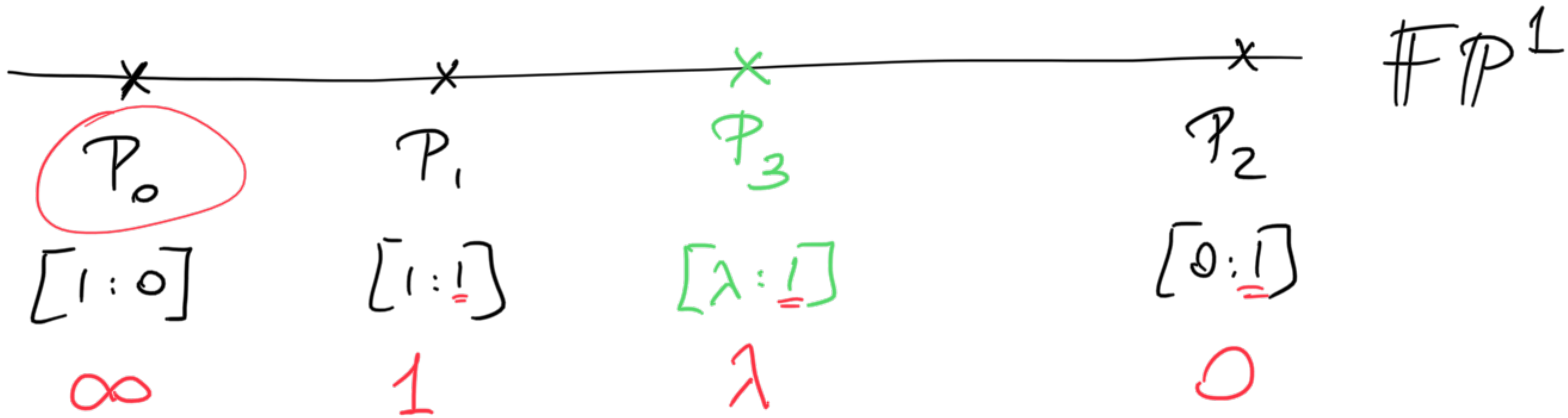
with $\lambda \in \mathbb{F} \setminus \{0,1\}$

and $(P_0 P_1 : P_2 P_3) = \underline{\lambda} \in \mathbb{F} \setminus \{0,1\}$.

Same for Q_i, in a different coordinate system.

$\exists$ unique τ taking $P_0, P_1, P_2 \to Q_0, Q_1, Q_2$ in this order. Now this takes $P_3 \to Q_3$ iff $\lambda_P = \lambda_Q$ iff $(P_0 P_1 : P_2 P_3) = (Q_0 Q_1 : Q_2 Q_3)$

$\square$



Once we fix P_0 to be "point at ∞" then <u>on the finite part</u> it makes sense to compare $\dfrac{P_2P_3}{P_2P_1} = \lambda$

This is (one interpretation of) the cross-ratio.

Summary

$$\left\{ \begin{matrix} \text{ordered triples of distinct} \\ (P_0, P_1, P_2) \in \mathbb{F}\mathbb{P}^1 \end{matrix} \right\} \Big/ PGL(\mathbb{F}^2) \longleftrightarrow \{*\}$$

$$\left\{ \begin{matrix} \text{ordered quadruples of distinct} \\ (P_0, P_1, P_2, P_3) \end{matrix} \right\} \Big/ PGL(\mathbb{F}^2) \longleftrightarrow \mathbb{F} \setminus \{0, 1\}$$

$$(P_0, P_1, P_2, P_3) \longmapsto (P_0 P_1 : P_2 P_3)$$

$$\shortparallel$$

$$\lambda \in \mathbb{F} \setminus \{0, 1\}$$

$\boxed{\text{Larger point sets} \rightsquigarrow \text{very interesting geometry!}}$