PROJECTIVE GEOMETRY LECTURE 6

AXIOMATIC PROJECTIVE GEOMETRY

Abstract Projective Plane $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where

- $\mathcal{P}$ is the set of points of Π,
- $\mathcal{L}$ is the set of lines of Π, and
- $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ is a relation called incidence:

for $p \in \mathcal{P}$, $L \in \mathcal{L}$, $(p, L) \in \mathcal{I}$ means

"point p lies on line L in Π"

we will denote $p \in L$.

We require a set of axioms for this abstract setup:

Basic axioms

A1: For all pairs of distinct points $p, q \in \mathcal{P}$, $p \neq q$, there exists a unique line $L \in \mathcal{L}$ with $p, q \in L$. i.e. $(p, L) \in \mathcal{I}$ and $(q, L) \in \mathcal{I}$.

A2: Any two lines $L_1, L_2 \in \mathcal{L}$ meet in at least one point i.e. $\exists\, p \in \mathcal{P}$ st. $p \in L_1$, $p \in L_2$.

A3: Any line contains at least 3 distinct points.

A4: There exist at least four points, of which no three are collinear.

Optional axioms

D: Desargues' theorem holds in Π.

P: Pappus' theorem holds in Π.

Two basic consequences of Basic Axioms

Consequence 1: Two distinct lines $L_1, L_2 \in \mathcal{L}$ meet in a unique point p (so we call $p = L_1 \cap L_2$).

Proof Assume $L_1 \neq L_2$, $L_1, L_2 \in \mathcal{L}$, and assume $p, q \in \mathcal{P}$ with $p, q \in L_1$, $p, q \in L_2$. Note that by A1, the line $\overline{pq}$ (if $p \neq q$) is unique! So we must have either $L_1 = L_2$ or else $p = q$!

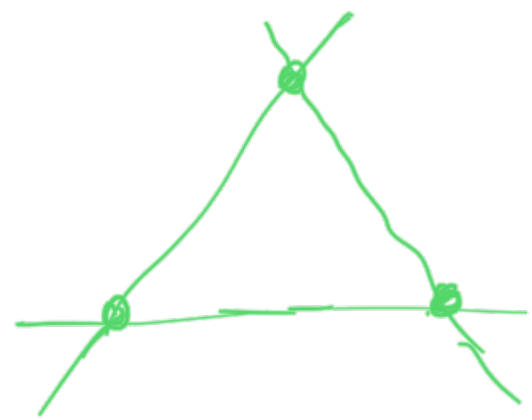
Note also that by A2, there is at least one common point $p \in L_1$, $p \in L_2$. We are done. $\square$

<u>Consequence 2</u>: Not all points $p \in \mathcal{P}$ are contained in a line.

<u>Proof</u> By A4, $\exists\, p_1, p_2, p_3, p_4 \in \mathcal{P}$, no three of which are collinear. So p_1, p_2, p_3 are not collinear. $\square$

"There exists a proper triangle in Π."

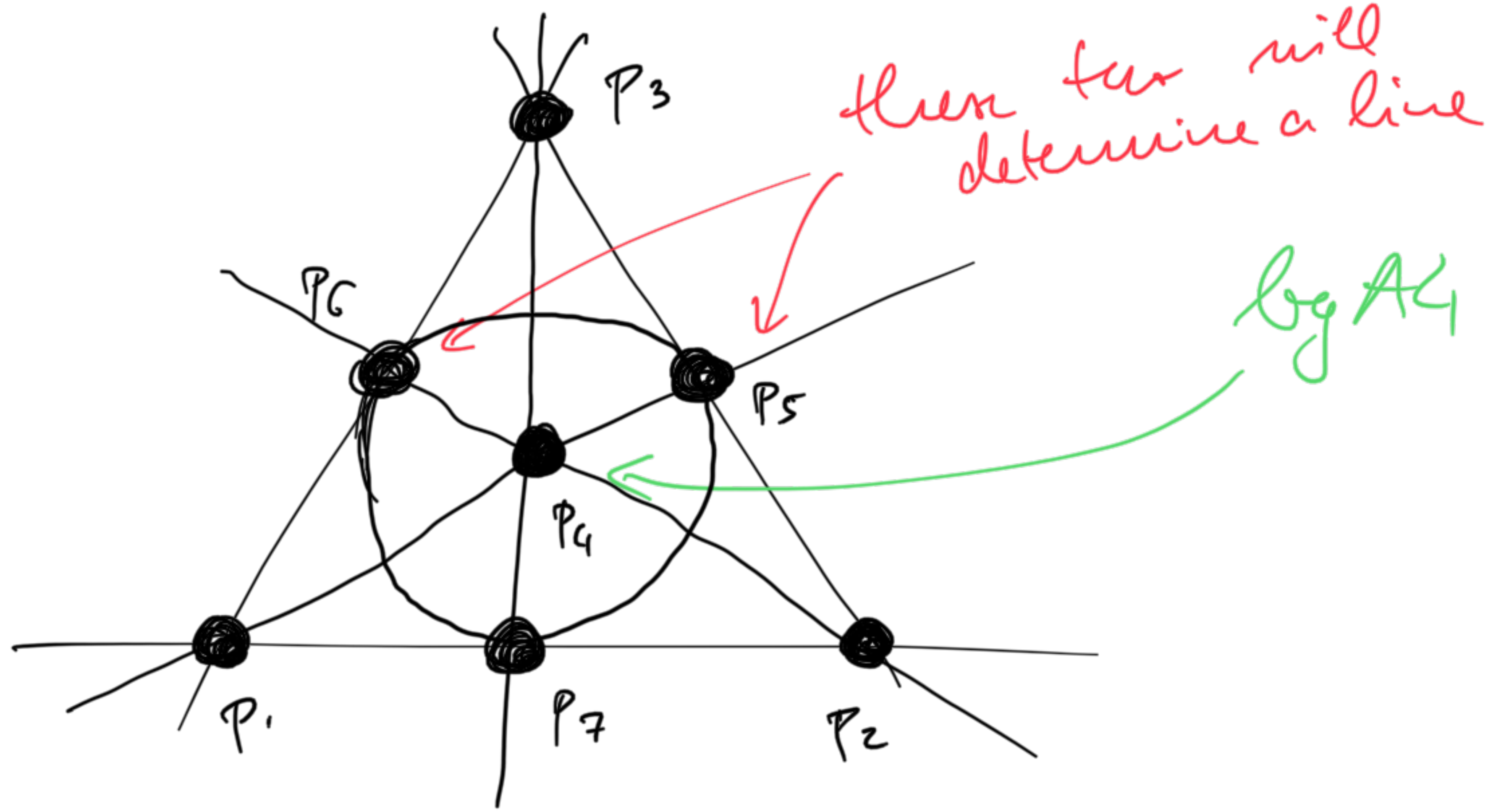
3 points, determining 3 different lines.



<u>Example</u>: it can be shown that the <u>smallest</u> example satisfying A1–A4 is the following:

$\mathcal{P}$ has seven elements

$\mathcal{L}$ has - - - - - - - .



FANO PLANE of 7 points and 7 lines

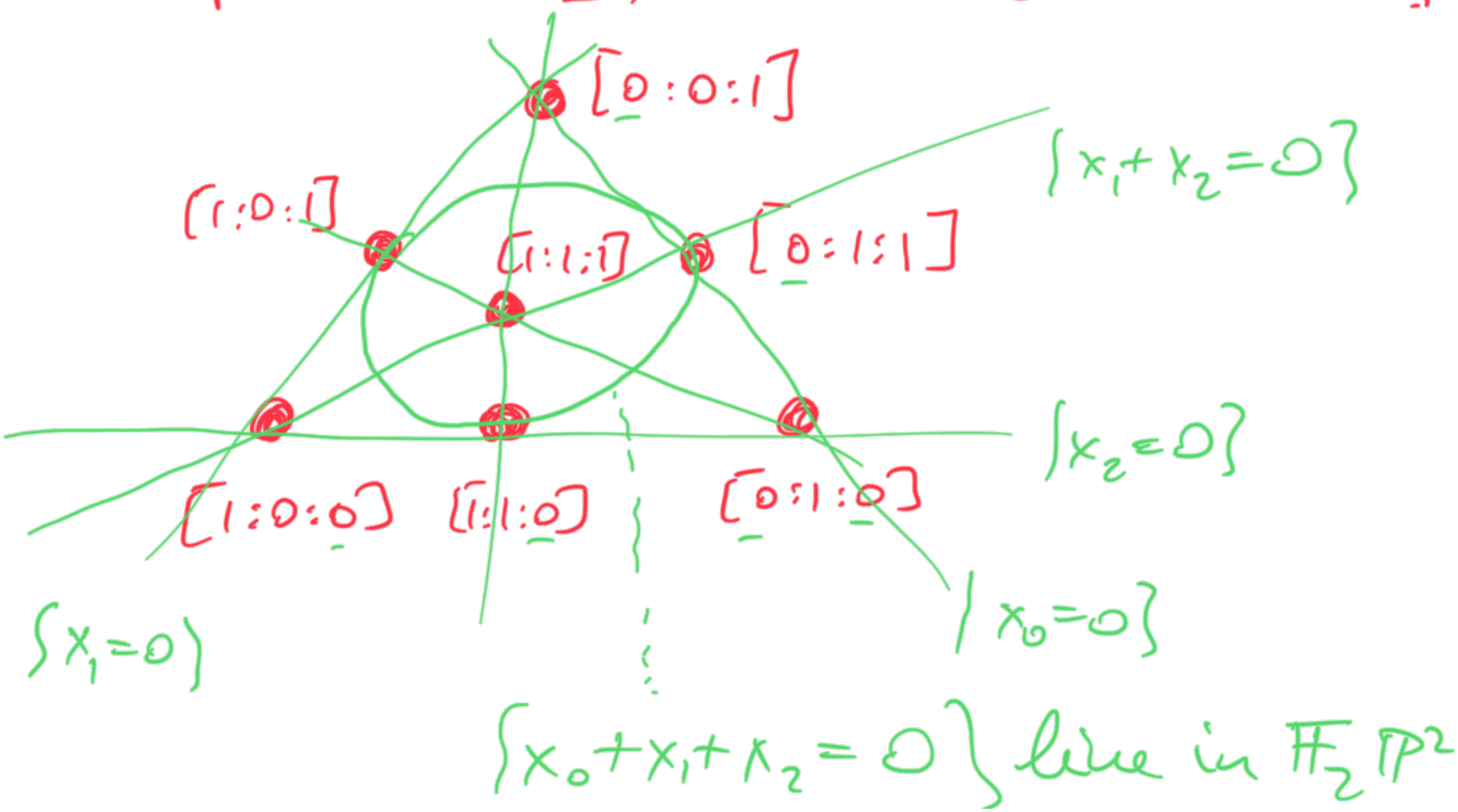
Example: $\mathbb{F}$ field, let Π to be the projective plane $\mathbb{F}\mathbb{P}^2 = \mathbb{P}(\mathbb{F}^3)$ with its usual set of lines & points, incidence. Then Π satisfies

A1 – A4 : as seen before

but also

D, P : since these are *theorems* on $\mathbb{F}\mathbb{P}^2$.

Note: the Fano plane is nothing but $\mathbb{F}_2\mathbb{P}^2$: the projective plane $\mathbb{P}(\mathbb{F}_2^3)$ over $\mathbb{F}_2 = \{0, 1, +, \cdot\}$



Question: Are these all the examples of projective planes? Given an abstract projective plane satisfying A1–A4, perhaps also D–P, is there a field $\mathbb{F}$ such that $\Pi \cong \mathbb{F}\mathbb{P}^2$

bijections on lines/points, preserving incidence

Short answer: A1–A4 in themselves do not guarantee this, but A1–A4–P do!

Theorem (Hessenberg's theorem)

A1 – A4 – P $\Longrightarrow$ D so basic axioms + Pappus implies Desargues, but not the other way round.

Main idea: given an abstract projective plane Π, how to start building a structure that will become (if we are lucky) the field $\mathbb{F}$?

Start with fixing line $L \in \mathcal{L}$, and a point $p \notin L$.

Write

$$\mathbb{P} = \{p \in L\} \cup \mathcal{A}, \quad O \in \mathcal{A}$$

points in the plane

points at ∞

finite points "affine plane" in Π

origin in the set of finite points

Def 1 A <u>collineation</u> is a pair of bijections

$\left\{ \begin{array}{l} \mathcal{P} \to \mathcal{P} \\ \mathcal{L} \to \mathcal{L} \end{array} \right\}$ preserving incidence relation, i.e.

an "automorphism of Π"

Def 2 A <u>central collineation</u> for (L, O) is a

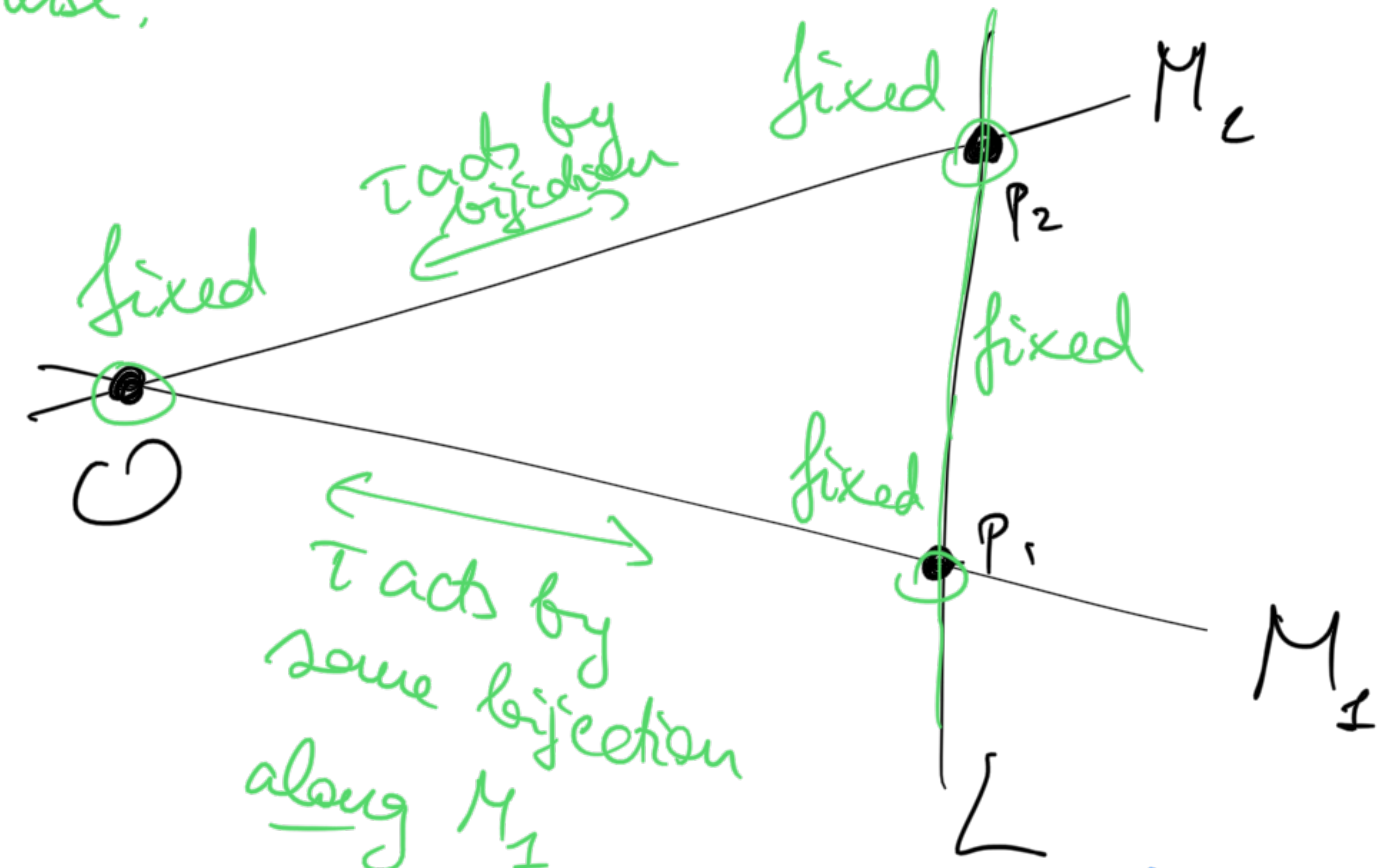
collineation $\tau : \begin{array}{l} \mathcal{P} \to \mathcal{P} \\ \mathcal{L} \to \mathcal{L} \end{array}$ such that

a.) $\tau(O) = O$

b.) $\forall p \in L$, $\tau(p) = p$. (Pointwise fixes L)

<u>It follows</u> that
- $\tau(L) = L$
- For all $M \ni O$, $\tau(M) = M$ (Fixes M)
- For all $q \in M \ni O$, $\tau(q) \in M$.
  (Not pointwise!)

Lines through O are mapped to themselves, not pointwise.



<u>Example</u> For $\Pi = \mathbb{F}\mathbb{P}^2$ over a field $\mathbb{F}$,

- projective transformations are examples of collineations (<u>FACT</u>: all collineations are projective transformations!)

• can choose $O = [1:0:0]$, $L = \{x_0 = 0\}$, $\mathbb{A} = \{x_0 \neq 0\}$.

Then an example of a central collineation is

$[x_0 : x_1 : x_2] \longmapsto [x_0 : \lambda x_1 : \lambda x_2]$ for $\boxed{\lambda \in \mathbb{F}^*}$

For if $x_0 = 0$ then $[0 : x_1 : x_2] = [0 : \lambda x_1 : \lambda x_2]$ for $\lambda \in \mathbb{F}^*$

and also $[1:0:0] \longmapsto [1:0:0]$ so O fixed also.

KEY FACT: these are all! Every central collineation is of this form.

Define $K = \{\tau \text{ central collineation for } (O, L)\} \cup \{\overset{\text{"zero"}}{0}\}$

This set carries a multiplication operation:

• "zero" $\cdot\ \tau$ = "zero"

- $\tau_1 \cdot \tau_2$: defined to be $\tau_1 \circ \tau_2$, the composite.

Note $(K, \cdot)$ is <u>associative</u>.

Note also that <u>K acts on $\mathcal{A}$</u>: $\tau \in K^*$ takes L to L pointwise, so $\tau|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. "zero" acts by mapping everything in $\mathcal{A}$ to the origin $\mathcal{O}$.

<u>If</u> $\Pi = \mathbb{F}\mathbb{P}^2$, central collineations as above, then these are
- multiplication on $\mathbb{F} = \mathbb{F}^* \cup \{0\}$
- standard action of $\mathbb{F}$ on $\mathcal{A} = \mathbb{F}^2$.

Finally, K <u>also carries</u> an <u>addition operation</u> defined by

- τ + "zero" = "zero"

- $\tau_1 + \tau_2$ defined geometrically (definition omitted, look at sources)

Theorem*** a.) $(K, +, \cdot)$ is associative and distributive, 0: "zero", 1: identity collineation

Enough to assume A1 – A4 + D

associativity of +

K is called a "skew field", $\cdot$ may not be commutative

b.) $\mathcal{A} \longleftrightarrow K^2$ and $\Pi \hookrightarrow K\mathbb{P}^2 = \mathbb{P}(K^3)$

$\mathcal{A} \circlearrowleft K$ $\qquad$ $K^2 \circlearrowleft$ multiplication action K

c.) $(K, \cdot)$ commutative $\iff$ P also holds.

So from $\underline{\underline{A1-A4}} + \underline{\underline{P}}$ (also D) we get that $K = F$ is a field, and $\Pi \overset{\sim}{\iff} F\mathbb{P}^2$.

STRONG LINK BETWEEN

- geometry of abstract projective planes
- algebra of fields and skew fields

There are also interesting, exotic Π's where <u>D also doesn't hold</u>: non-Desarguesian projective planes.