Projective Geometry

Lecture 7: Duality

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Duality on projective planes

Recall **abstract projective plane** $\Pi = \{\mathcal{P}, \mathcal{L}, \mathcal{I}\}$

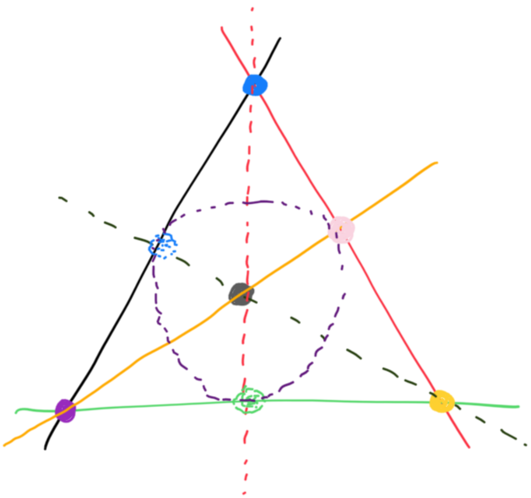
- $\mathcal{P}$ is the set of points;
- $\mathcal{L}$ is the set of lines;
- $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ is the incidence relation between points and lines;
- subject to some axioms.

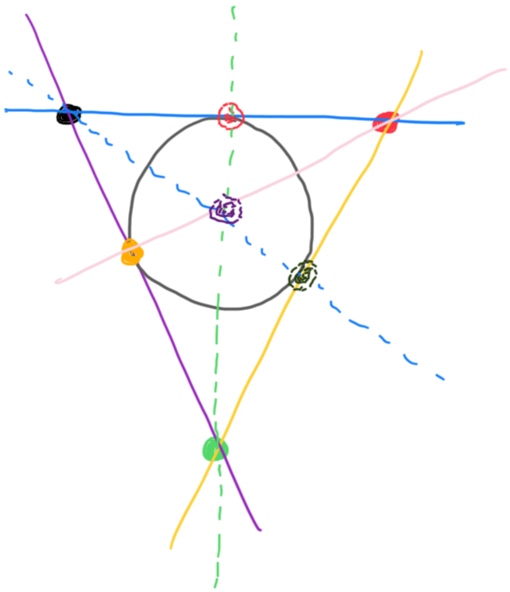
Duality: Following Hilbert, we can think instead of $\mathcal{L}$ as the set of points; $\mathcal{P}$ as the set of lines; sometimes the axioms will remain the same!

This will allow us to deduce new theorems from old, dualizing an earlier statement.
Example: Desargues' Theorem (see later).

To extend duality to higher dimensions, linear algebra will again be very helpful.

Example: duality on the Fano plane





A little bit of Part A Linear Algebra

We recall that to any vector space V over a field $\mathbb{F}$ we can associate the **dual space**

$$V^* = \{f : V \to \mathbb{F} \text{ linear}\}.$$

If $\dim V = n$, then V and V^* are isomorphic, since V^* also has dimension n. However, this isomorphism depends on a choice of basis.

Recall that if $\{e_1, \dots e_n\}$ is a basis of V, then a basis of V^* is given by the **dual basis** $\{E_1, \dots, E_n\}$, defined by

$$E_i(e_j) = \delta_{ij}$$

and extended linearly.

The double dual V^{**}, that is, the dual of V^*, **is canonically** isomorphic to V. Explicitly, the map

$$\begin{aligned} \varphi :\ V &\to V^{**} \\ v &\mapsto (f \mapsto f(v) \text{ for } f \in V^*) \end{aligned}$$

defines an isomorphism between V and V^{**}.

A little bit more Part A Linear Algebra

Further, given a linear subspace $U \leq V$, we have its annihilator

$$U^\circ = \{f \in V^* : f(u) = 0 \text{ for all } u \in U\}.$$

Proposition For subspaces U, U_1, U_2 of a finite-dimensional vector space V

(i) if $U_1 \leq U_2$, then $U_2^\circ \leq U_1^\circ$; that is, taking the annihilator reverses inclusion;

(ii) $(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ$;

(iii) $(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ$;

(iv) $\dim U + \dim U^\circ = \dim V$;

(v) $(U^\circ)^\circ = \varphi(U)$.

Conclusion: get an inclusion-reversing one-to-one correspondence $U \leftrightarrow U^\circ$ between linear subspaces of V and linear subspaces of V^*.

Note: We shall use the canonical isomorphism φ to identify spaces with their double duals, and subspaces with their double annihilators, without further comment.

Principle of Duality

With $\dim V = n+1$, consider n-dimensional projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$.

Principle of Duality There is an **inclusion-reversing correspondence**

$$\{\text{linear subspaces } \mathbb{P}(U) \subset \mathbb{P}(V)\} \longleftrightarrow \{\text{linear subspaces } \mathbb{P}(U^\circ) \subset \mathbb{P}(V^*)\}.$$

By the dimension formula, if $\mathbb{P}(U)$ is an m-dimensional linear subspace of $\mathbb{P}^n = \mathbb{P}(V)$, then...
...U has dimension $m+1$
...so U° has dimension $(n+1)-(m+1) = n-m$
...and hence $\mathbb{P}(U^\circ)$ is a linear subspace of $\mathbb{P}(V^*)$ of dimension $n-m-1$.

From the Proposition above, we also get the following properties of this:

$$\langle \mathbb{P}(U_1), \mathbb{P}(U_2) \rangle^\circ = \mathbb{P}(U_1^\circ) \cap \mathbb{P}(U_2^\circ)$$

$$(\mathbb{P}(U_1) \cap \mathbb{P}(U_2))^\circ = \langle \mathbb{P}(U_1^\circ), \mathbb{P}(U_2^\circ) \rangle.$$

Duality of points and hyperplanes

Points of $\mathbb{P}(V^*)$ represent 1-dimensional subspaces of V^*. These correspond to hyperplanes in $\mathbb{P}(V)$, which represent n-dimensional subspaces of V.

The correspondence assigns to $\langle f \rangle$, where $f \in V^* - \{0\}$, the hyperplane $\mathbb{P}(\ker(f))$ in $\mathbb{P}(V)$.

In homogeneous coordinates, the point $[a_0 : \ldots : a_n]$ in the dual projective space $\mathbb{P}(V^*)$ corresponds to the hyperplane

$$\{a_0x_0 + \ldots + a_nx_n = 0\} \subset \mathbb{P}(V).$$

Note that scaling all the a_i does not alter the hyperplane.

Conversely, hyperplanes in $\mathbb{P}(V^*)$ correspond to points in $\mathbb{P}(V^{**})$ and thus to points in $\mathbb{P}(V)$.

Duality of points and lines in the projective plane

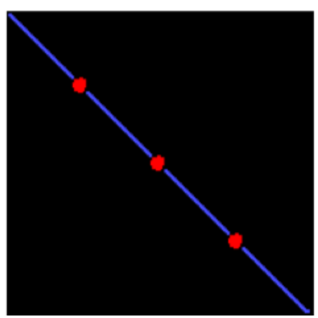
Assume $\dim V = 3$ so $\dim \mathbb{P}(V) = 2$.

Duality interchanges points of $\mathbb{P}(V) = \mathbb{FP}^2$ and lines in $\mathbb{P}(V^*) = \mathbb{FP}^2$.

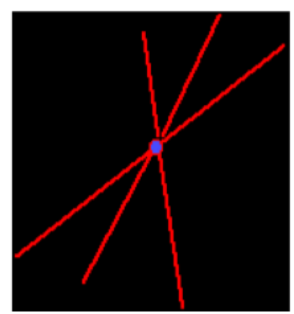
If $P = [p], Q = [q]$ are two distinct points on the line $L = \mathbb{P}U \subset \mathbb{P}(V)$ with $U = \langle p, q \rangle$, then the lines $\mathbb{P}\langle p \rangle^\circ, \mathbb{P}\langle q \rangle^\circ$ meet at the point $\mathbb{P}U^\circ$ of $\mathbb{P}(V^*)$.

More generally, a set of collinear points in $\mathbb{P}(V)$ corresponds under duality to a set of **concurrent** lines in $\mathbb{P}(V^*)$ (lines passing through a common point).

three collinear points



three concurrent lines



Points and lines in general position in the projective plane

On a projective plane $\mathbb{P}(V) = \mathbb{FP}^2$ we can define four **lines** to be in general position if no three of them are concurrent.

This is equivalent to the four points they represent in $\mathbb{P}(V^*)$ being in general position.

Under duality, a line

$$\{\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0\} \subset \mathbb{P}(V)$$

corresponds to the point

$$[\alpha_0 \colon \alpha_1 \colon \alpha_2] \in \mathbb{P}(V^*).$$

So by the General Position Theorem, four lines in $\mathbb{P}(V)$ which are in general position can be assumed to have the equations

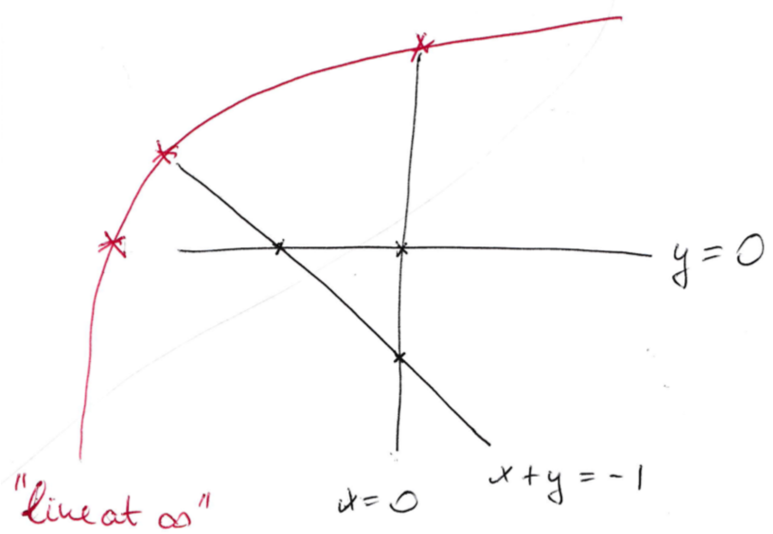
$$x_0 = 0, \qquad x_1 = 0, \qquad x_2 = 0, \qquad x_0 + x_1 + x_2 = 0.$$

Lines in general position in the projective plane

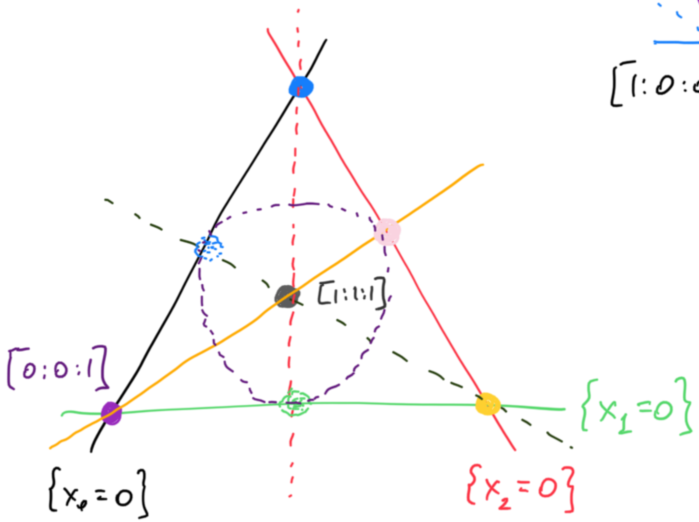
Four lines in $\mathbb{P}(V)$ which are in general position can be assumed to have the equations

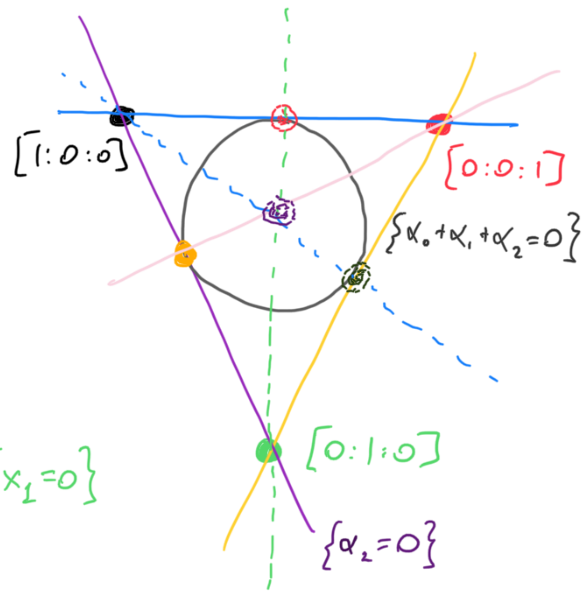
$$x_0 = 0, \qquad x_1 = 0, \qquad x_2 = 0, \qquad x_0 + x_1 + x_2 = 0.$$

In affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, we get the following picture.



Duality on the Fano plane $\mathbb{F}_2\mathbb{P}^2$ in coordinates





The dual of Desargues's Theorem

The dual of Desargues's Theorem in the plane is as follows.

Theorem Let $\pi, \alpha, \alpha', \beta, \beta', \gamma, \gamma'$ be seven distinct lines in a projective plane $\mathbb{FP}^2$ over a field $\mathbb{F}$, such that the points $\alpha \cap \alpha', \beta \cap \beta'$ and $\gamma \cap \gamma'$ are distinct and all lie on π.
Then the lines joining $\alpha \cap \beta, \alpha' \cap \beta'$ and $\beta \cap \gamma, \beta' \cap \gamma'$ and $\gamma \cap \alpha, \gamma' \cap \alpha'$ are concurrent.

The Principle of Duality says that we do not need to prove this result separately: it follows from the original result on the dual plane!