Projective Geometry

Lecture 8: Conics

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Real conics in the plane $\mathbb{R}^2$

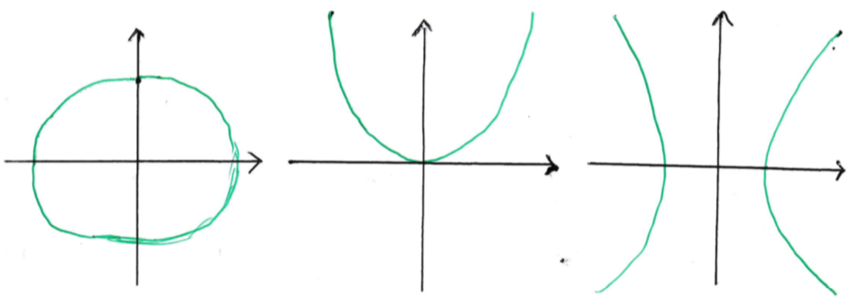
A **conic** is a plane curve given by a quadratic equation.

A real (affine) conic is a curve $C \subset \mathbb{R}^2$ given by a quadratic equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

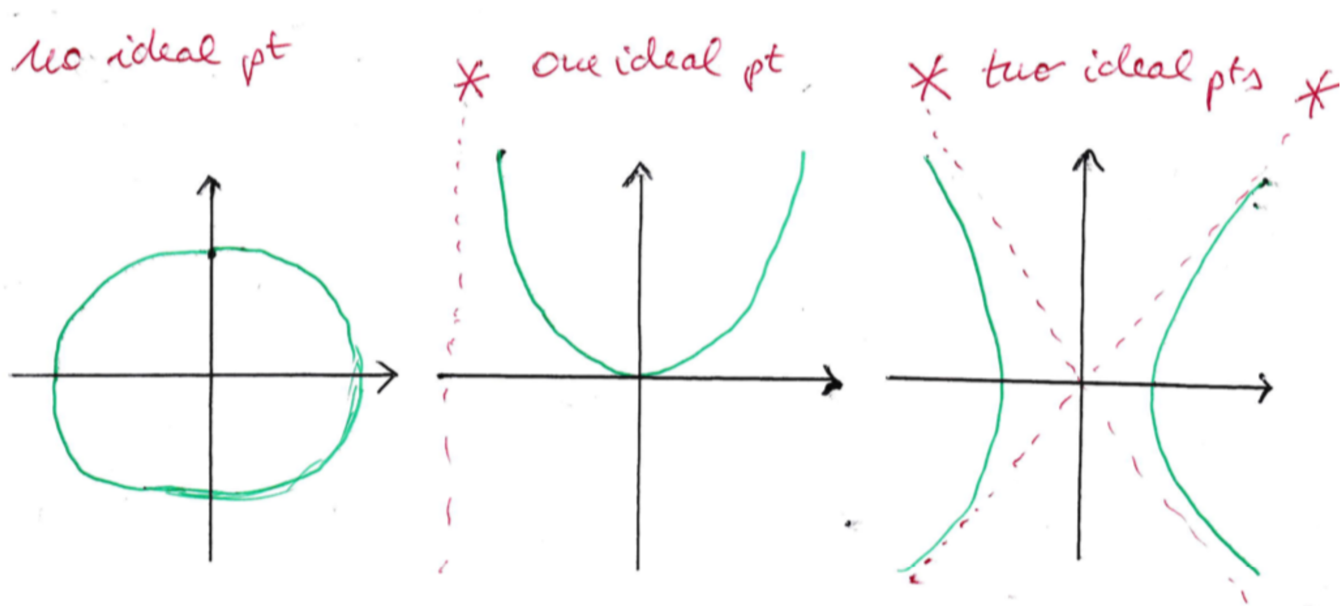
Three types of conics: **ellipse, parabola, hyperbola**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad y = ax^2 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Real conics in the plane $\mathbb{R}^2$ and their asymptotes

Where do these conics meet the **ideal line**?



Real conics: extending to the projective plane

Consider the projective plane $\mathbb{RP}^2$ with coordinates $[x_0 : x_1 : x_2]$, its ideal line $L_\infty = [0 : x_1 : x_2]$ and the ordinary plane $\mathbb{R}^2 = [1 : x : y]$.

The transformation between projective coodinates $[x_0 : x_1 : x_2]$ and affine coordinates (x, y) is $x = x_1/x_0$, $y = x_2/x_0$.

We get the projective equations

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = x_0^2 \qquad\qquad x_0 x_2 = a x_1^2 \qquad\qquad \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = x_0^2$$

The real projective solutions with $x_0 = 0$ (ideal points) are

$$\emptyset \qquad\qquad [0 : 0 : 1] \qquad\qquad [0 : a : \pm b]$$

Note that, after a linear change coordinates, all three equations have the form

$$-y_0^2 + y_1^2 + y_2^2 = 0.$$

Bilinear forms over arbitrary fields

A **symmetric bilinear form** on a vector space V over $\mathbb{F}$ is a map

$$B : V \times V \to \mathbb{F}$$

such that

(i) $B(v, w) = B(w, v)$;

(ii) B is linear in v (and hence, by (i), in w).

If an addition we have that

(iii) if $B(v, w) = 0$ for all w, then $v = 0$,

then we say the form is **nondegenerate** or **nonsingular**.

Remark Note that the conditions are different from those seen in other contexts. Over $\mathbb{F} = \mathbb{R}$, we could require positive definiteness instead of (iii). Over $\mathbb{F} = \mathbb{C}$, we could require sesquilinearity instead of (i)-(ii). Our conditions make sense for any field.

Bilinear forms over arbitrary fields in coordinates

If we choose a basis $\{e_0, \ldots, e_n\}$ of V, then a bilinear form is given by

$$B(v,w) = v^t X w$$

for a symmetric matrix X given by

$$X_{ij} = B(e_i, e_j).$$

Nondegeneracy of the form is equivalent to nonsingularity (invertibility) of the matrix X.

A bilinear form is determined (if the characteristic of $\mathbb{F}$ is $\neq 2$), by the associated **quadratic form**

$$Q(v) = B(v,v),$$

for we can recover B via the polarisation identity

$$B(v,w) = \frac{1}{4}(B(v+w,v+w) - B(v-w,v-w)) = \frac{1}{4}(Q(v+w) - Q(v-w))$$

Projective quadrics and conics over arbitrary fields

A **projective quadric** is the locus of points

$$C = \{[v] \colon Q(v) = 0\} \subset \mathbb{P}(V)$$

where $v \mapsto Q(v) = B(v, v)$ is a (not identically zero) quadratic form on V.

In matrix form, we get

$$C = \{[v] \colon v^t X v = 0\} \subset \mathbb{P}(V)$$

where X is a nonzero symmetric matrix over F.

A **projective conic** is a projective quadric in a projective plane $\mathbb{P}(V) = \mathbb{FP}^2$

$$C = \{[v] \colon Q(v) = 0\} \subset \mathbb{FP}^2$$

where $v \mapsto Q(v) = B(v, v)$ is a (not identically zero) quadratic form on a three-dimensional vector space V.

More on projective quadrics and conics

Projective transformations send quadrics to quadrics. If we write the quadratic form in terms of a symmetric matrix X, then its image under a projective transformation is the form defined by the symmetric matrix $M^t X M$, where M defines the projective transformation.

Note also that if the quadratic forms Q and Q' are proportional, that is $Q'(v) = \lambda Q(v)$ for all v, then they define the same quadric.

Definition We say a quadric C is **nonsingular**, if the associated symmetric bilinear form B is nondegenerate.

Projective quadrics: some examples

Example 1 The simplest example is the identity matrix $X = I$, for which we get the quadric

$$C_1 = \left\{ [x_0 \colon x_1 \colon \ldots \colon x_n] \,\middle|\, \sum_{i=0}^{n} x_i^2 = 0 \right\} \subset \mathbb{FP}^n.$$

Note that in the most familiar example $\mathbb{F} = \mathbb{R}$, this quadric is **empty**.

However, it has plenty of points over other fields such as $\mathbb{F} = \mathbb{C}$.

Extended exercise What happens over $\mathbb{F} = \mathbb{F}_p$?

Projective quadrics: some examples

Example 2 Consider the matrix

$$X = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We get the quadric

$$C_2 = \left\{ [x_0 \colon x_1 \colon \ \dots \colon x_n] \,\middle|\, -x_0^2 + \sum_{i=1}^{n} x_i^2 = 0 \right\} \subset \mathbb{FP}^n.$$

For $\mathbb{F} = \mathbb{R}$, in affine coordinates x_i/x_0, this would give back the equation of the sphere in $\mathbb{R}^n$. So this has plenty of points over $\mathbb{F} = \mathbb{R}$.

Note that for $n = 2$, all the familiar plane conics were projectively equivalent to this conic.

Projective quadrics: some examples

Example 3 Let $n = 2$ and consider the matrix

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We get the plane conic

$$C_3 = \left\{ [x_0 \colon x_1 \colon x_2] \,\middle|\, x_1^2 - x_2^2 = 0 \right\} \subset \mathbb{FP}^2.$$

Note that the equation now **factorizes**:

$$C_3 = \{ [x_0 \colon x_1 \colon x_2] \,|\, (x_1 - x_2)(x_1 + x_2) = 0 \} \subset \mathbb{FP}^2.$$

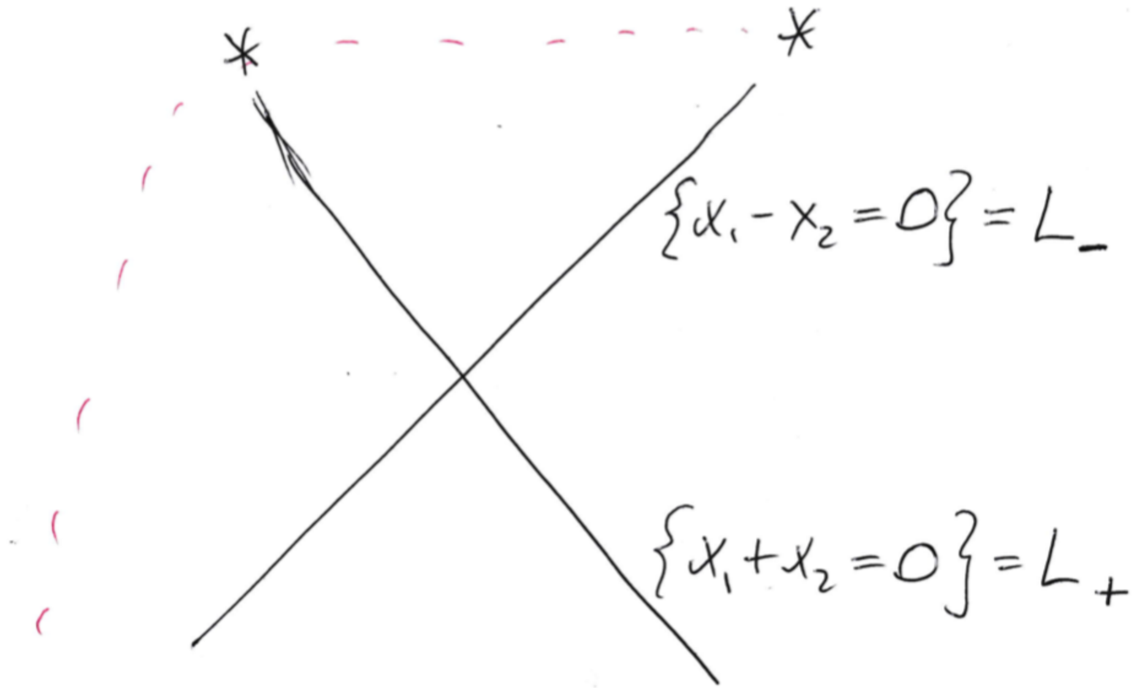
So we can write

$$C_3 = L_- \cup L_+ \subset \mathbb{FP}^2,$$

a union of the lines $L_\pm = \{x_1 \pm x_2 = 0\} \subset \mathbb{FP}^2$. These lines meet at the point $[1 : 0 : 0] \in \mathbb{FP}^2$.

Projective quadrics: some examples

$$C_3 = \left\{[x_0 \colon x_1 \colon x_2] \,\middle|\, x_1^2 - x_2^2 = 0\right\} = L_- \cup L_+ \subset \mathbb{FP}^2.$$



Nonsingular quadrics

We say that a quadric is **nonsingular**, if the associated bilinear form B is nondegenerate, equivalently the matrix X is invertible.

A **singular point** of a quadric

$$C = \{[v]\colon Q(v) = 0\} \subset \mathbb{P}(V)$$

is $[v] \in \mathbb{P}(V)$ for nonzero v such that $B(v, w) = 0$ for all $w \in V$, equivalently nonzero $v \in \ker X$.

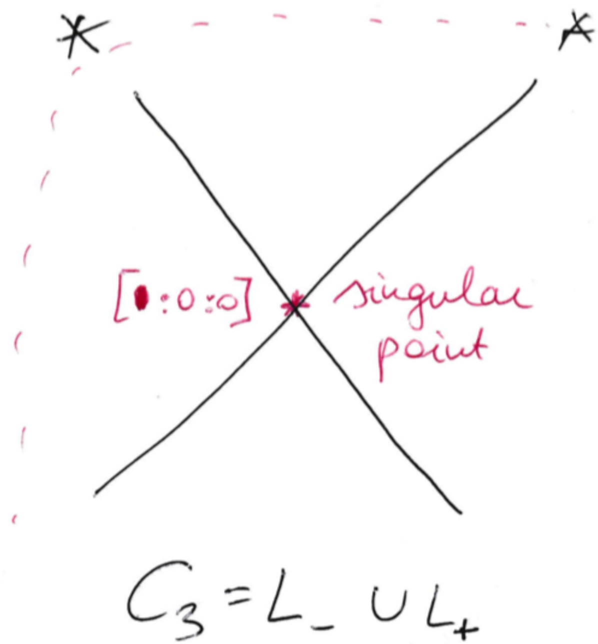
The quadrics C_1, C_2 in Examples 1-2 above are nonsingular.

The plane conic $C_3 = L_- \cup L_+$ is singular at $[1 : 0 : 0] = L_- \cap L_+$.

Remark The terminology comes from thinking about conics as **submanifolds** (locally) of Euclidean space (see parallel Part A course). At a nonsingular point, a quadric is a submanifold. At singular points, the Jacobian condition fails; hence the name.

Singular and nonsingular projective conics

$$C_2 = \{-x_0^2 + x_1^2 + x_2^2 = 0\} \qquad C_3 = \{x_1^2 - x_2^2 = 0\} = L_- \cup L_+$$



Diagonalizing quadratic forms

Over the fields $\mathbb{R}$ and $\mathbb{C}$, we can diagonalise quadratic forms.

Theorem Let $v \mapsto Q(v) = B(v, v)$ be a quadratic form defined on an $(n+1)$-dimensional vector space V.

(i) Over the base field $\mathbb{F} = \mathbb{C}$, there is a basis $\{e_0, \ldots, e_n\}$ of V, with respect to which

$$Q(v) = \lambda_0^2 + \ldots + \lambda_r^2,$$

where $v = \sum_{i=0}^n \lambda_i e_i$.

(ii) Over the base field $\mathbb{F} = \mathbb{R}$, there is a basis $\{e_0, \ldots, e_n\}$ of V, with respect to which

$$Q(v) = \lambda_0^2 + \ldots + \lambda_r^2 - \lambda_{r+1}^2 - \ldots - \lambda_{r+s}^2,$$

where $v = \sum_{i=0}^n \lambda_i e_i$.

Remark Note that Q is non-degenerate if and only if $r = n$, respectively $r + s = n$. More generally, $\mathrm{rk}\ X = r + 1$, $\mathrm{rk}\ X = r + s + 1$ in the two cases.

Proof of the diagonalization theorem: Step 1

Write

$$Q(v) = v^t X v = \sum_{i,j} X_{ij} v_i v_j$$

in some basis, where X is a nonzero, symmetric matrix.

Step 1 We can assume that some X_{ii} is nonzero. For if all $X_{ii} = 0$, find a nonzero X_{ij}. Introduce new variables

$$y_i = \frac{1}{2}(v_i + v_j), \quad y_j = \frac{1}{2}(v_i - v_j).$$

Now Q has the term

$$X_{ij} v_i v_j = X_{ij} y_i^2 - X_{ij} y_j^2$$

with nonzero diagonal terms in the new basis.

Proof of the diagonalization theorem: Steps 2-3

Step 2

Now we complete the square.

$$\frac{1}{X_{ii}}\left(\sum_j X_{ij}v_j\right)^2 = X_{ii}v_i^2 + 2\sum_{j\neq i} X_{ij}v_jv_i + \text{terms in } v_j \quad (j \neq i)$$

so by introducing the new variable $y_i = \sum X_{ij}v_j$, we can put Q into the form

$$Q(v) = \frac{1}{X_{ii}}y_i^2 + Q'(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$$

for some quadratic form Q' with one less variable.

Step 3

Now we repeat the process until we have diagonalised Q. Finally rescaling the variables appropriately, we can assume that we have the stated forms; note that over $\mathbb{R}$, we cannot change the sign of y_i^2 by rescaling.

Classification of complex projective quadrics

Turning to quadrics, for the field $\mathbb{F} = \mathbb{C}$, our diagonalization theorem says that every complex projective quadric in $\mathbb{CP}^n$ is projectively equivalent to

$$D_r = \left\{ [x_0 \colon x_1 \colon \dots \colon x_n] \,\middle|\, \sum_{i=0}^{r} x_i^2 = 0 \right\} \subset \mathbb{CP}^n.$$

for some $0 \leq r \leq n$.

The quadric D_r is nonsingular if and only if $r = n$.

Classification of complex projective conics

Specializing to $n = 2$, we get that every complex projective conic is projectively equivalent to one of the following:

(i) The **nonsingular conic** $D_2 = \left\{[x_0 : x_1 : x_2] \,\middle|\, x_0^2 + x_1^2 + x_2^2 = 0\right\} \subset \mathbb{CP}^2$.

(ii) The **line pair** $D_1 = \left\{[x_0 : x_1 : x_2] \,\middle|\, x_0^2 + x_1^2 = 0\right\} \subset \mathbb{CP}^2$.

Indeed, as before, we can write

$$D_1 = M_+ \cup M_- \subset \mathbb{CP}^2$$

for

$$M_\pm = \{x_0 \pm i x_1 = 0\} \subset \mathbb{CP}^2.$$

(iii) The **double line** $D_0 = \left\{[x_0 : x_1 : x_2] \,\middle|\, x_0^2 = 0\right\} \subset \mathbb{CP}^2$.

Indeed, this is “twice the line” M, where

$$M = \{x_0 = 0\} \subset \mathbb{CP}^2.$$

Classification of nonsingular real projective conics

Let us now see what we get in the case of conics with $\mathbb{F} = \mathbb{R}$, restricting to the nonsingular case. (Exercise: think about the singular cases!)

Writing $v = \sum x_i e_i$, there are four nonsingular quadratic forms to consider.

(i) $Q_1(v) = x_0^2 + x_1^2 + x_2^2$.

(ii) $Q_2(v) = x_0^2 + x_1^2 - x_2^2$.

(iii) $Q_3(v) = x_0^2 - x_1^2 - x_2^2$.

(iv) $Q_4(v) = -x_0^2 - x_1^2 - x_2^2$.

The conics corresponding to Q_1 and Q_4 are **empty**. The conics corresponding to Q_2, Q_3 are **the same**, as the forms are constant multiples of each other (up to change of variables).

Hence indeed, up to projective equivalence **there is a unique nonempty nonsingular real projective conic**

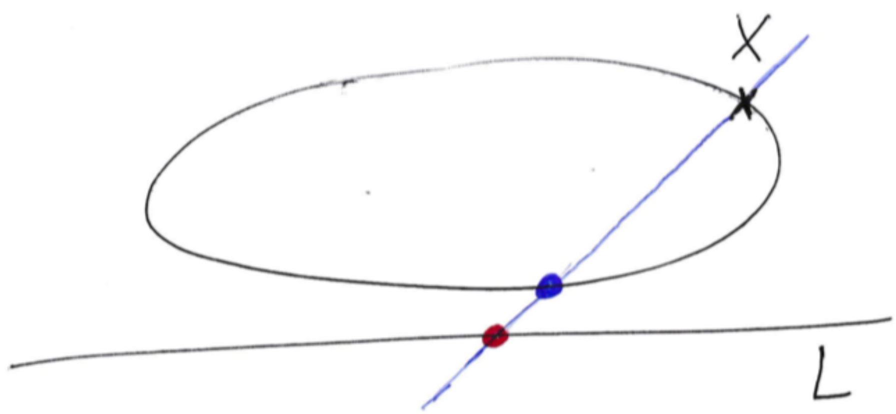
$$C = \left\{-x_0^2 + x_1^2 + x_2^2 = 0\right\} \subset \mathbb{RP}^2.$$

Parametrising conics

Work over an arbitrary field $\mathbb{F}$ again, only assuming char $\mathbb{F} \neq 2$.

Consider a nonsingular conic $C \subset \mathbb{FP}^2$, a point $X \in C$, and a projective line $L \subset \mathbb{FP}^2$ not containing X.

We will give a description of C using the following geometric idea: projection from X onto the line L sets up a bijection between the conic C and the line L.

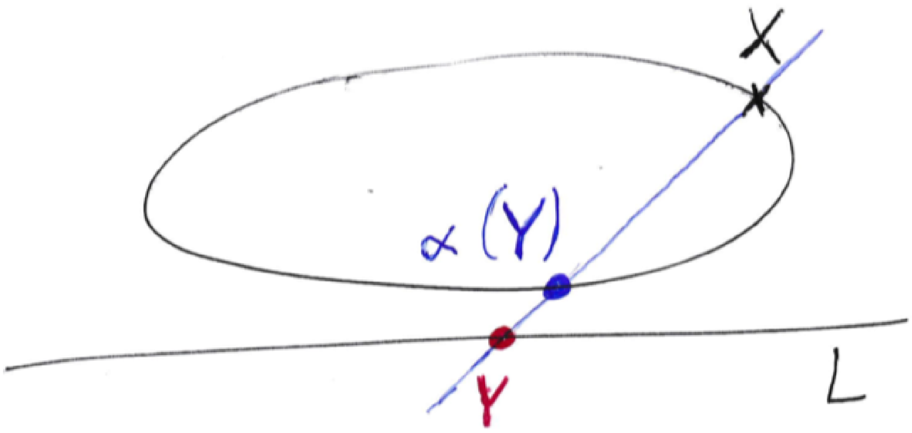


Parametrising conics: the theorem

Theorem Let C be a nonsingular conic in the projective plane $\mathbb{P}(V) = \mathbb{F}\mathbb{P}^2$, over a field $\mathbb{F}$ with char $\mathbb{F} \neq 2$. Let X be a point of C. Let $L = \mathbb{P}(U)$ be a projective line in the plane not containing X. Then there is a bijection

$$\alpha : L \to C$$

such that $X, Y, \alpha(Y)$ are collinear for each $Y \in L$.



Parametrising conics: the proof

Proof Let B denote the nondegenerate bilinear form whose quadratic form Q defines the conic C. Let $X = [x]$ be a point on C, so that $B(x, x) = 0$.

For each $Y \in \mathbb{P}(U)$, we want to see where (other than at X) the projective line XY meets the conic. We will find that there is a unique such point and this will be $\alpha(Y)$.

Let $Y \in \mathbb{P}(U)$ have representative vector $y \in U$, so that x, y are linearly independent, as we are assuming $X \notin L = \mathbb{P}(U)$.

Consider the 2-dimensional subspace $W_y = \langle x, y \rangle \subset V$, so the projective line we are considering is $XY = \mathbb{P}(W_y)$.

Key Claim: The bilinear form B cannot be identically zero on the space W_y.

Parametrising conics: proof of the Key Claim

Suppose that the bilinear form B is identically zero on the two-dimensional subspace $W_y \subset V$.
Let $v \in V \setminus W_y$. Consider

$$\langle v \rangle^\circ = \{w \in V : B(v, w) = 0\}.$$

Then as B is nonsingular, this is a single linear condition on $w \in V$, so

$$\dim \langle v \rangle^\circ = 2.$$

Hence by the Dimension of Intersection formula inside the 3-dimensional vector space V, we get

$$\dim \left(\langle v \rangle^\circ \cap W_y \right) \geq 1.$$

Pick any nonzero vector $t \in \langle v \rangle^\circ \cap W_y$, then t annihilates v and also W_y under B.
But then t annihilates $\langle v, W_y \rangle = V$, which is impossible as B is non-degenerate.

Parametrising conics: the proof

With respect to the basis $\{x, y\}$, the form Q restricted to W_y is

$$Q(\lambda_0 x + \lambda_1 y) = 2\lambda_0\lambda_1 B(x, y) + \lambda_1^2 B(y, y).$$

$B(x, y), B(y, y)$ are not both zero by the Key Claim. So the projective line $\mathbb{P}(W_y)$ meets the conic C at two points, corresponding to the solutions $[\lambda_0 : \lambda_1]$.

One intersection point is the basepoint $X = [x]$, corresponding to

$$[\lambda_0 : \lambda_1] = [1 : 0].$$

Defined $\alpha(Y)$ to be the other intersection point, corresponding to

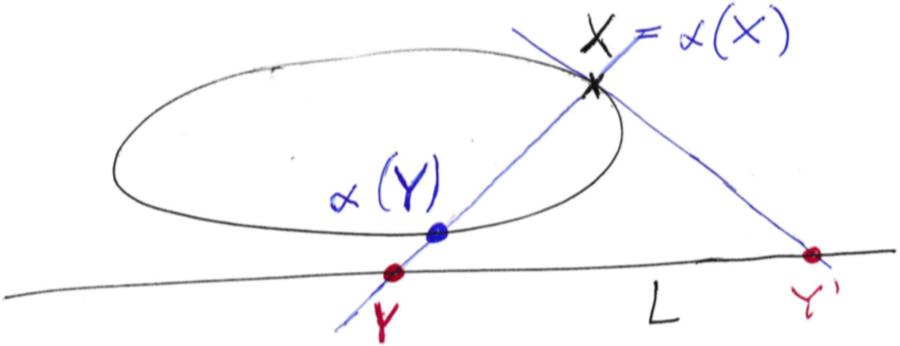
$$[\lambda_0 : \lambda_1] = [B(y, y) : -2B(x, y)].$$

So $X, Y, \alpha(Y)$ are collinear by construction.

α is bijective: given any point $Z \neq X$ on the conic, the projective line XZ meets the line L in a unique point Y, and then $\alpha(Y) = Z$.

Parametrising conics: the proof

For $Z = X$ itself, the image $\alpha(X) = X$, coming from the intersection point Y' of the **tangent line** at X with the line L.



This corresponds to the case then the quadratic above has a double root. (Think about this!)