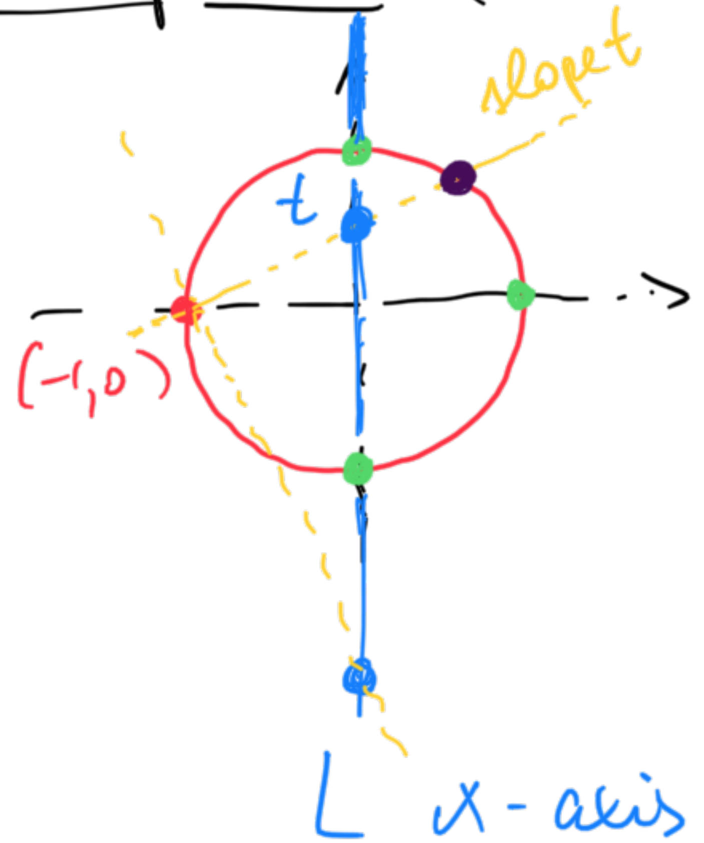
Projective Geometry Lecture 9

Concrete example: "parametrize the unit circle over $\mathbb{F}$"

$$C = \{ x^2 + y^2 = 1 \} \subseteq \mathbb{F}^2 \; ; \quad \text{char } \mathbb{F} \neq 2 \; ; \; p = (-1,0) \in C$$

Real picture $(\mathbb{F} = \mathbb{R})$



$p \in C$

$(0,t) \in L$

Consider line through $p \in C$ and $(0,t) \in L$, this has equation

$$y = tx + t$$

Next, find the "other" intersection point between yellow line and C.

We get a set of equations $\begin{cases} y = tx + t \\ x^2 + y^2 = 1 \end{cases}$

from which we get

$$x^2 + (tx+t)^2 = 1$$

(root at -1)

$$(t^2+1)x^2 + 2t^2x + (t^2-1) = 0$$ - quadratic for x.

$$x_{1,2} = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^2+1)(t^2-1)}}{2(t^2+1)} = \begin{cases} -1 & \text{red pt} \\ \dfrac{1-t^2}{1+t^2} & \text{purple pt} \end{cases}$$

$y = tx + t$, so get two intersection pts $(-1, 0)$ and $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

Our theorem or common thought: we get a bijection $t \in \mathbb{F} \Longleftrightarrow \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right) \in C \setminus p$

$t = \infty \Longleftrightarrow p$, tangent line

"vertical slope" to C at $(-1,0) = p$.

Conclusion: one-to-one correspondence

$$\mathbb{F} \cup \{\infty\} \Longleftrightarrow C$$

over an arbitrary $\mathbb{F}$ (char $\mathbb{F} \neq 2$).

Aside: parametrize circle $/\mathbb{R}$

$S^1 = \{x^2+y^2=1\} \subset \mathbb{R}^2$

• $(x,y) = (\cos\theta, \sin\theta)$ "angle parametrization"

$S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ "trig functions are periodic"

"topology"

• $y = \sqrt{1-x^2}$: bijection $[-1,1) \longleftrightarrow$ "upper semicircle"

$\longleftrightarrow$ "lower semicircle"

"analysis"

• $(x,y) = \left(\dfrac{1-t^2}{1+t^2}, \dfrac{2t}{1+t^2}\right)$ "algebraic/geometric parametrization"

simple functions : bijection $\mathbb{R} \longleftrightarrow S^1 \setminus \{pt\}$

$\mathbb{RP}^1 \longleftrightarrow S^1$

Over any $\mathbb{F}$, move to projective coordinates

$$C_{proj} = \{ x_1^2 + x_2^2 = x_0^2 \} \subset \mathbb{F}\mathbb{P}^2$$

$$\begin{array}{c} \cup \\ [1:-1:0] = p \end{array}$$

Parameter $t \in \mathbb{F}$ gets replaced by $[a:b] \in \mathbb{F}\mathbb{P}^1$ so that $t = b/a$

$[a:b] = [1:b/a]$ $(a \neq 0)$

Get parametrization $\frac{x_1}{x_0} = \frac{1-t^2}{1+t^2} = \frac{a^2-b^2}{a^2+b^2}$

$$\frac{x_2}{x_0} = \frac{2t}{1+t^2} = \frac{2ab}{a^2+b^2} \qquad \text{so}$$

we get

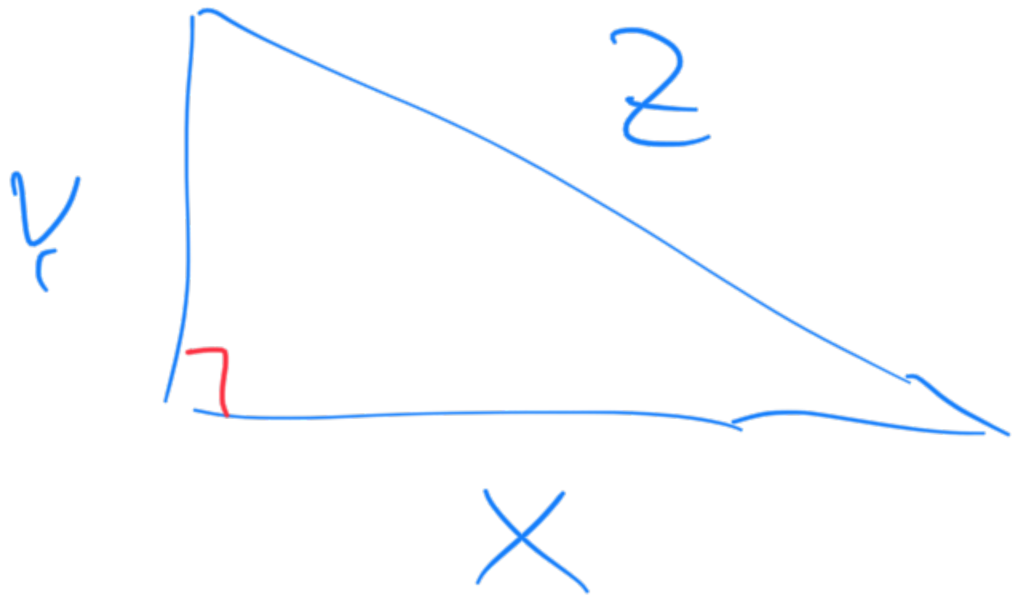
$$\boxed{\begin{array}{ccc} \mathbb{FP}^1 & \longleftrightarrow & C_{proj} \subset \mathbb{FP}^2 \\ [a:b] & \longmapsto & [(a^2+b^2):(a^2-b^2):2ab] \end{array}}$$

Exercise: do the calculation following exact steps from last lecture to get this!

Application: to solving Diophantine equations

i.e. equations which require integer solutions.

Equation of Pythagoras: find right-angled Δs with integer (positive) side lengths!



$(*)\quad X^2+Y^2=Z^2$ with $X,Y,Z\in\mathbb{N}$

Notice: integer solutions come from rational solutions!

If I can find $(X,Y,Z)\in\mathbb{Q}^3$ satisfying $(*)$ then

- clear denominators
- fix signs

to get $(X,Y,Z)\in\mathbb{Z}^3$

Also $\quad X^2+Y^2=Z^2$ gives us

$\left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right)^2 = 1$ so a point of the circle over $\mathbb{F} = \mathbb{Q}$.

We have parametrized $C_{\mathbb{Q}}$ by our formula, so any $\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ is of the form $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ for $t = \frac{n}{m} \in \mathbb{Q}$, from Theorem + Calculation.

So $\left(\frac{X}{Z}, \frac{Y}{Z}\right) = \left(\frac{m^2-n^2}{m^2+n^2}, \frac{2mn}{m^2+n^2}\right)$ for $(m,n) \in \mathbb{Z}$

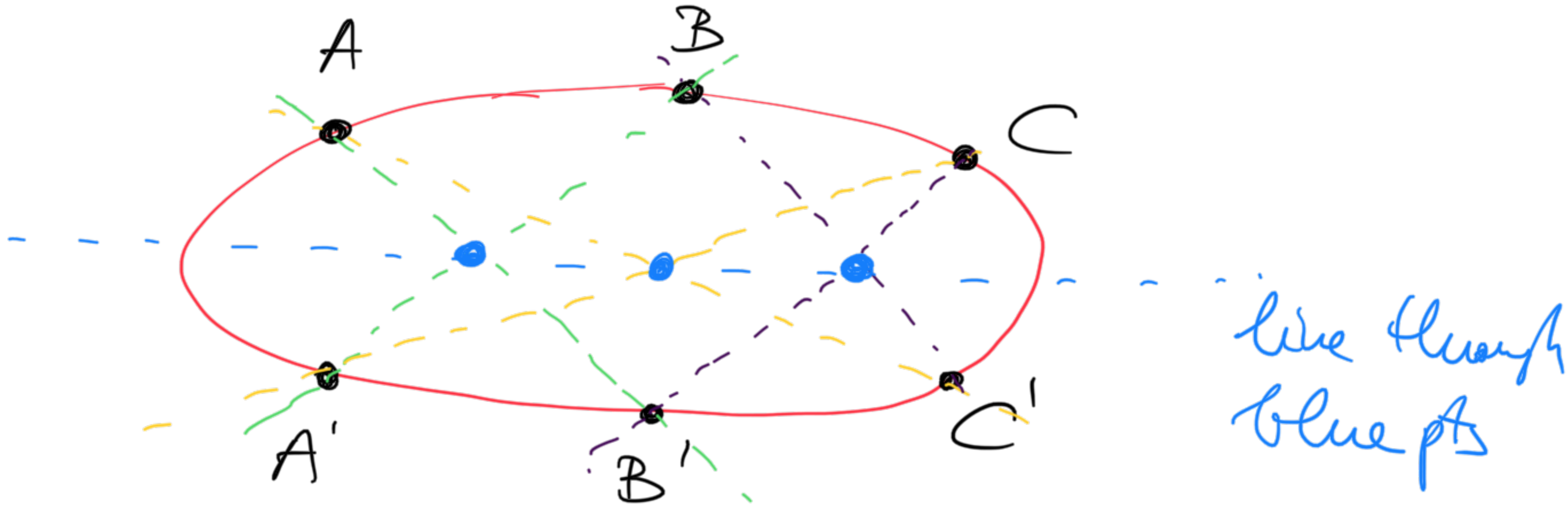
i.e. $\boxed{(X, Y, Z) = (m^2 - n^2, 2mn, m^2+n^2)}$

This method applies to other quadratic diophantine equations ... see problem sheet and past papers.

- recognise quadric C
- find a point p on C
- use parametrization construction to find all points on C $(\mathbb{F} = \mathbb{Q})$

Pascal's Theorem Let $\mathcal{C}$ be a non-degenerate conic in $\mathbb{F}\mathbb{P}^2$. Let (A, B, C) and (A', B', C') be two triples of distinct points on $\mathcal{C}$. Then

$(AB' \cap A'B)$, $(AC' \cap A'C)$, $(BC' \cap B'C)$ are collinear.



<u>Remark 1</u> We know line pair is a degenerate conic. For a line pair $\mathcal{C} = L_1 \cup L_2$, this becomes Pappus' theorem.

<u>Remark 2</u> Over $\mathbb{F} = \mathbb{R}$

$$\begin{array}{ccc} \text{non-degenerate conic} & \xleftrightarrow[PGL(3)]{\text{proj. equivalent}} & \text{unit circle} \subseteq \mathbb{RP}^2 \\ \mathcal{C} \subset \mathbb{RP}^2 \text{ (with a pt)} & & \cap \\ & & \mathbb{R}^2 \end{array}$$

The assumptions and conclusion are unchanged by a projective transformation.

<u>Prop</u> Pascal's theorem over $\mathbb{R}$ $\Longleftrightarrow$ Pascal's theorem for unit circle

This has elementary proof based on angles, triangles, sine theorem...

<u>Where does this subject go?</u>

- Algebraic geometry: objects in $\mathbb{FP}^n$ defined by polynomial equations (homogeneous) of <u>higher degree</u>

<u>Here</u> $\deg = 1$, $\deg = 2$ (quadrics / conics)

- Number theory: study equations in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, number fields (finite extensions of $\mathbb{Q}$)

- Applied math / Theoretical physics: eg. Penrose's <u>twistor theory</u> fundamentally uses ideas of projective geometry

- Geometry: manifolds in projective space (compact)