Groups

ASO course Trinity 2021

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Example sheet 1

1. Let $K \trianglelefteq G$ and let $\bar{H} \leq G/K$. Let $\pi : G \to G/K$ denote the quotient map $g \mapsto gK$. Show that

$$H = \pi^{-1}(\bar{H}) = \{g \in G : gK \in \bar{H}\}$$

is a subgroup of G, containing K as a normal subgroup, with $H/K = \bar{H}$. Show further that if $\bar{H} \trianglelefteq G/K$ then $H \trianglelefteq G$.

2. Identify the following groups from their presentation

(i) $G_1 = \langle x \mid x^6 = 1\rangle$,

(ii) $G_2 = \langle x, y \mid xy = yx\rangle$,

(iii) $G_3 = \langle x, y \mid x^3y = y^2x^2 = x^2y\rangle$,

(iv) $G_4 = \langle x, y \mid xy = yx, x^5 = y^3\rangle$,

(v) $G_5 = \langle x, y \mid xy = yx, x^4 = y^2\rangle$.

[For G_4 you may wish to consider the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ given by $(a, b) \mapsto 3a + 5b$].

3. Let $G = \langle x, y \mid x^2 = y^2 = 1\rangle$.

(i) Show that G is infinite

(ii) Let $z = xy$. Show that every element of G can be uniquely written as z^k or yz^k where k is an integer.

(iii) Deduce that G is isomorphic to the *infinite dihedral group*

$$D_\infty = \langle y, z | y^2 = 1, yzy^{-1} = z^{-1}\}.$$

(iv) By considering appropriate reflections and translations, show that G may be identified with the isometry group of the integers $\mathbb{Z}$, considered as a subset of the real line with the Euclidean metric.

4. Let G be a non-Abelian group of order 8.

(i) Show that G has an element a of order 4.

(ii) Let $A = \langle a\rangle$ and let $b \in G - A$. Show that $bab^{-1} = a^{-1}$ and either $b^2 = 1$ or $b^2 = a^2$.

(iii) Deduce that there are, up to isomorphism, exactly two non-Abelian groups of order 8, and five groups of order 8 in total.

Show that one of the non-Abelian groups may be identified with the *quaternion group*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\},$$

where we have the usual quaternionic relations

$$i^2 = j^2 = k^2 = -1 \quad : \quad ij = k = -ji.$$

5. Write down all possible composition series of the following groups and verify the Jordan-Hölder Theorem for them:

$$C_{12}, \quad D_{10}, \quad D_8, \quad Q_8.$$

6. Let H and K be subgroups of a group G. Show that

$$HK = \{hk : h \in H, \ k \in K\}$$

is a subgroup of G if and only if $HK = KH$.

7. Show that $(\mathbb{Q}, +)$ is not finitely generated.