*As in lectures, $(m, n)$ denotes the highest common factor of $m$ and $n$.*

**1.** Let $a$ be a positive integer and suppose that in its decimal expansion it has 7 digits: $a = a_0 + 10a_1 + \cdots + 1000000a_6$. Show that $a$ is divisible by 7 if and only if $a_0 + 3a_1 + 2a_2 - a_3 - 3a_4 - 2a_5 + a_6$ is divisible by 7.

**2.** Find a positive integer $x$ such that $x \equiv 3 (\bmod 4)$, $2x \equiv 5 (\bmod 9)$ and $7x \equiv 1 (\bmod 11)$.

**3.** Find the smallest positive integer $x$ such that $x \equiv 11 (\bmod 59)$ and $x \equiv 29 (\bmod 71)$.

**4.** Show that $2^{340} \equiv 1 (\bmod 341)$. Comment on this in connection with Fermat's Little Theorem.

**5.** Let $n = (6t+1)(12t+1)(18t+1)$ with $6t+1$, $12t+1$ and $18t+1$ all prime numbers. Prove that

$$a^{n-1} \equiv 1 (\bmod n)$$

whenever $(a, n) = 1$. Comment on this in connection with Fermat's Little Theorem.

**6.** Show that if $x$ is an integer then $x^{10} \in \{-1, 0, 1\} (\bmod 25)$.

**7.** For which $N$ is the following true: if you take an $N$ digit number, reverse its digits and then add the result to the original number, you always get a multiple of 11?

**8.** Find all primes $p$ for which the map $\phi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ defined by $\phi(x) = x^{13}$ is a group homomorphism.

**9.** Find all four-digit numbers $N$ such that, when written in decimal, the last four digits of any power of $N$ are the same as the digits of $N$.

**10.** For each of the following properties, show that there are infinitely many positive integers $n$ which do *not* have that property.

(i) $n$ is the sum of at most 3 squares;

(ii) $n$ is the sum of at most 8 sixth powers;

(iii) $n$ is the sum of at most 11 tenth powers;

(iv) $n$ is the sum of at most 15 fourth powers;

(v) $n$ is the sum of at most 7 (positive) seventh powers.

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