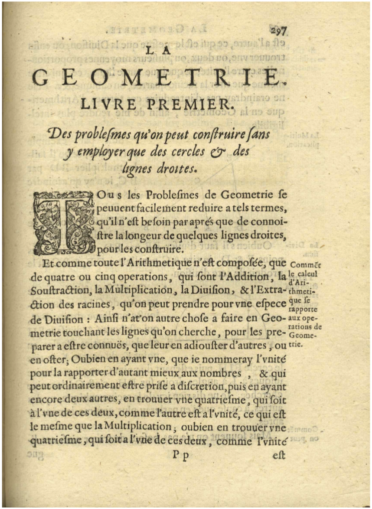
BO1 History of Mathematics
Lecture III
Analytic geometry and the beginnings of calculus
Part 3: Geometry and tangents

MT 2020 Week 2

Analytic (algebraic) geometry



LA

GEOMETRIE.

LIVRE PREMIER.

Des probleſmes qu'on peut conſtruire ſans y employer que des cercles & des lignes droites.

TOus les Probleſmes de Geometrie ſe peuuent facilement reduire a tels termes, qu'il n'eſt beſoin par aprés que de connoiſtre la longeur de quelques lignes droites, pour les conſtruire.

Et comme toute l'Arithmetique n'eſt compoſée, que de quatre ou cinq operations, qui ſont l'Addition, la Souſtraction, la Multiplication, la Diuiſion, & l'Extraction des racines, qu'on peut prendre pour vne eſpece de Diuiſion : Ainſi n'a-t'on autre choſe a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a eſtre connuës, que leur en adiouſter d'autres, ou en oſter; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement eſtre priſe a diſcretion, puis en ayant encore deux autres, en trouuer vne quatrieſme, qui ſoit à l'vne de ces deux, comme l'autre eſt a l'vnité, ce qui eſt le meſme que la Multiplication; oubien en trouuer vne quatrieſme, qui ſoit a l'vne de ces deux, comme l'vnité eſt

Commēt le calcul d'Arithmetique ſe rapporte aux operations de Geometrie.

Pp

La géométrie (1637)

Solution of geometric problems by algebraic methods

Appendix to
Discours de la méthode

"by commencing with objects the simplest and easiest to know, I might ascend by little and little"

Descartes' analytic geometry

We may label lines (line segments) with letters a, b, c, ...

Then $a+b$, $a-b$, ab, a/b, $\sqrt{a}$ may be constructed by ruler and compass.

Descartes' method

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For examples, see Katz (brief), §10.2, or Katz (3rd ed.), §14.2

Algebraic methods in geometry: some objections

Pierre de Fermat (1656, France):

> *I do not know why he has preferred this method with algebraic notation to the older way which is both more convincing and more elegant ...*

Thomas Hobbes (1656, England):

> *... a scab of symbols ...*

The beginnings of calculus: tangent methods

Calculus:

- finding tangents;
- finding areas.

Descartes' method for finding tangents (1637)

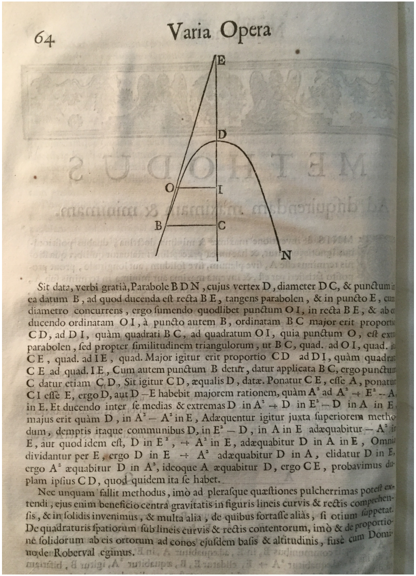
- based on finding a circle that touches the curve at the given point — a tangent to the circle is then a tangent to the curve
- used his algebraic approach geometry to find double roots to equation of intersection
- was in principle a general method — but laborious

Fermat's method for finding tangents

Pierre de Fermat (1601–1665):

- steeped in classical mathematics
- like Descartes, investigated problems of Pappus
- devised a tangent method (1629) quite different from that of Descartes

Fermat's tangent method (1629)



64 Varia Opera

E
D
O I
B C
N

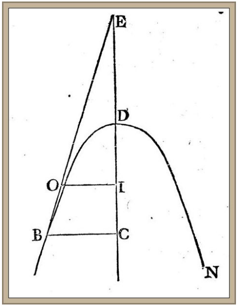
Sit data, verbi gratià, Parabole B D N, cujus vertex D, diameter D C, & punctum in ea datum B, ad quod ducenda est recta B E, tangens parabolen, & in puncto E, cum diametro concurrens, ergo sumendo quodlibet punctum O I, in recta B E, & ab ea ducendo ordinatam O I, à puncto autem B, ordinatam B C major erit proporti C D, ad D I, quàm quadrati B C, ad quadratum O I, quia punctum O, est ext parabolen, sed propter similitudinem triangulorum, ut B C, quad. ad O I, quad. i C E, quad. ad I E, quad. Major igitur erit proportio C D ad D I, quàm quadra C E ad quad. I E, Cum autem punctum B detur, datur applicata B C, ergo punctum C datur etiam C D, Sit igitur C D, æqualis D, datæ. Ponatur C E, esse A, ponatur C I esse E, ergo D, aut D – E habebit majorem rationem, quàm A^2 ad $A^2 + E^2 - A$, in E. Et ducendo inter se medias & extremas D in A^2 + D in E^2 – D in A in E, majus erit quàm D, in A^2 – A^2 in E, Adæquentur igitur juxta superiorem methodum, demptis itaque communibus D, in E^2 – D, in A in E adæquabitur – A^2 in E, aut quod idem est, D in E^2, + A^2 in E, adæquabitur D in A in E, Omnia dividantur per E, ergo D in E + A^2 adæquabitur D in A, elidatur D in E, ergo A^2 æquabitur D in A^2, ideoque A æquabitur D, ergo C E, probavimus duplam ipsius C D, quod quidem ita se habet.

Nec unquam fallit methodus, imò ad plerasque quæstiones pulcherrimas potest extendi, ejus enim beneficio centra gravitatis in figuris lineis curvis & rectis comprehensis, & in solidis invenimus, & multa alia, de quibus fortasse aliàs, si otium suppetat. De quadraturis spatiorum sub lineis curvis & rectis contentorum, imò & de proportione solidorum ab eis ortorum ad conos ejusdem basis & altitudinis, fusè cum Domino de Roberval egimus.

Worked out c. 1629, but only published posthumously in *Varia opera mathematica*, 1679.

See *Mathematics emerging*, §3.1.1.

Fermat's tangent method (1629)



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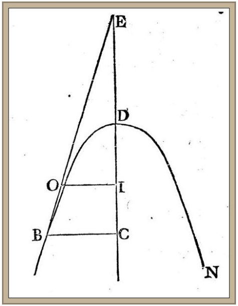

Choose an arbitrary point B on the parabola.

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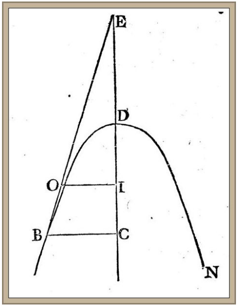

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Suppose that the tangent at B exists, and that it crosses the axis of the parabola at E.

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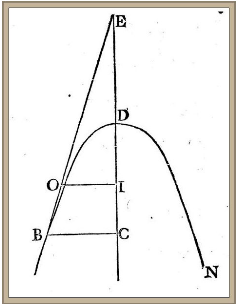

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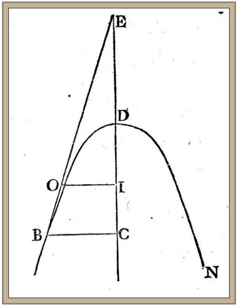

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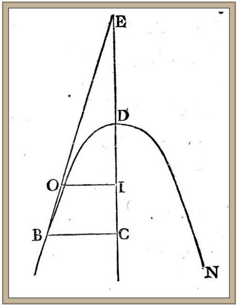

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Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

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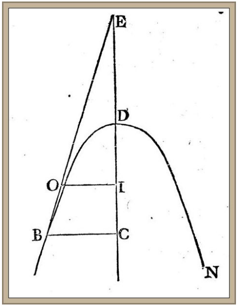

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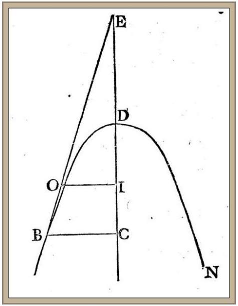

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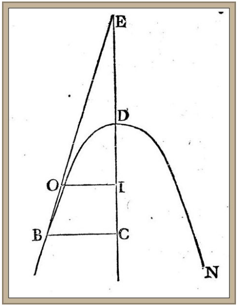

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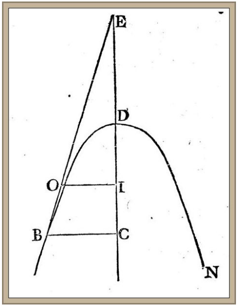

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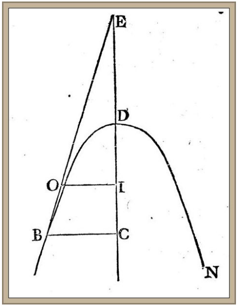

Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$

Put $CD = d$, $CE = a$, $CI = e$, so that

$$\frac{d}{d-e} > \frac{a^2}{(a-e)^2}.$$

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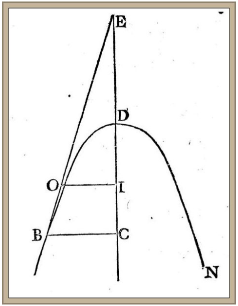

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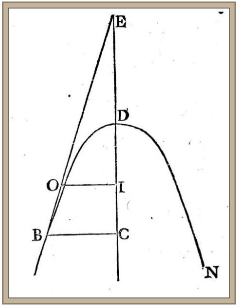
Now (Fermat says), we obtain equality as e decreases (as OI becomes BC):

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Fermat's tangent method (1629)

We solve the equality



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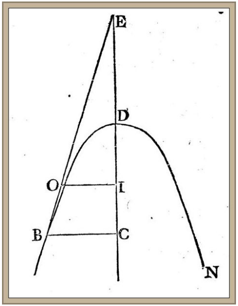

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$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Rearranging gives $\quad de^2 + a^2e = 2ade.$

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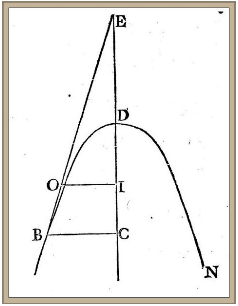

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Cancel e: $de + a^2 = 2ad.$

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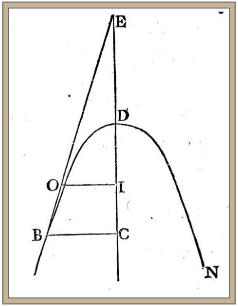

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Now e will be small, so we can neglect it, leaving us with $a^2 = 2ad.$

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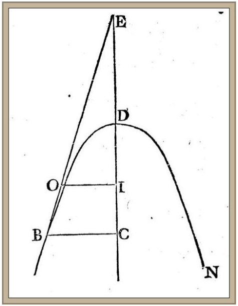

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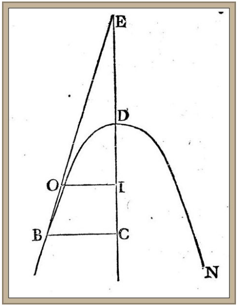

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Or $CE = 2 \times CD.$