# BO1 History of Mathematics
# Lecture IV
# The beginnings of calculus, continued
# Part 1: Quadrature

MT 2020 Week 2

# Summary

Part 1

- *Enri*: a non-Western prelude
- Quadrature (finding areas)

Part 2

- Indivisibles
- Infinitesimals

Part 3

- The contributions of Newton & Leibniz

# Seki and *enri*

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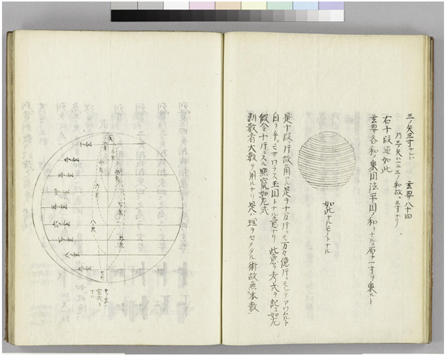

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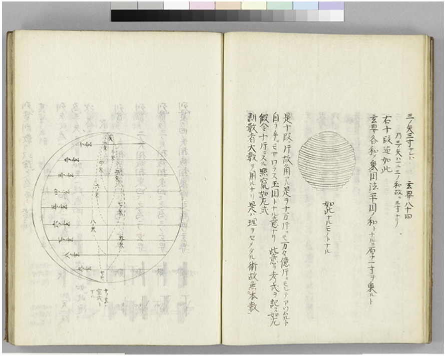

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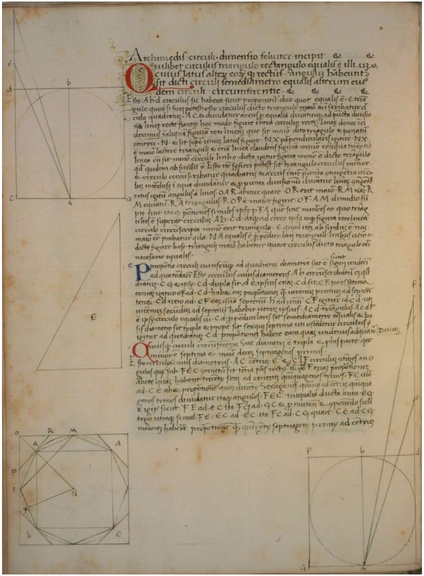
But *enri* was much narrower in scope than calculus

# Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



Translated into Latin as *Dimensio circoli* by Jacobus Cremonensis, c. 1450–1460

Illustrated by Piero della Francesca

Available online with other texts by Archimedes

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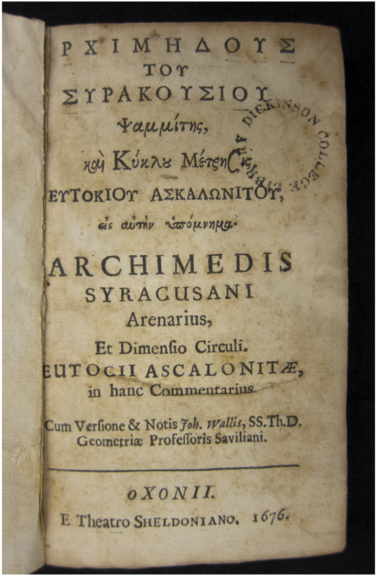


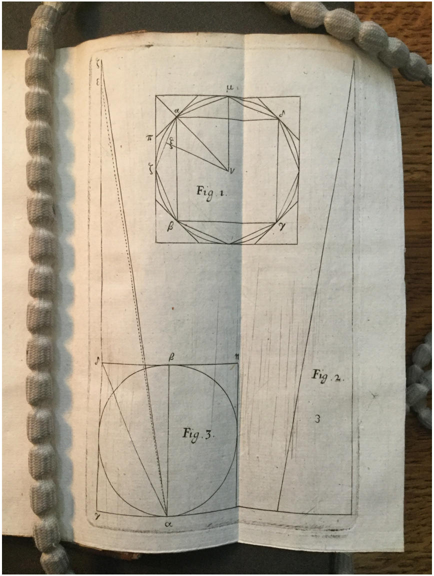


Edition by John Wallis, Oxford, 1676

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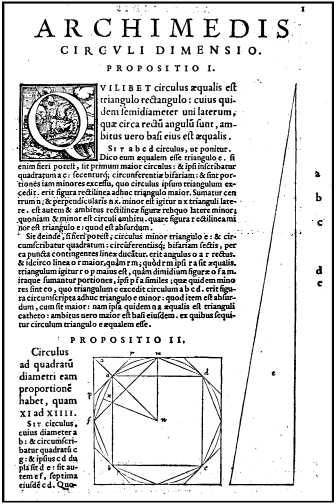

1

ARCHIMEDIS

CIRCVLI DIMENSIO.

PROPOSITIO I.

QVILIBET circulus æqualis est triangulo rectangulo: cuius quidem semidiameter uni laterum, quæ circa rectũ angulũ sunt, ambitus uero basi eius est æqualis.

SIT a b c d circulus, ut ponitur. Dico eum æqualem esse triangulo e. si enim fieri potest, sit primum maior circulus: & ipsi inscribatur quadratum a c: secenturq; circunferentiæ bifariam: & sint portiones iam minores excessu, quo circulus ipsum triangulum excedit. erit figura rectilinea adhuc triangulo maior. Sumatur centrum n; & perpendicularis n x. minor est igitur n x trianguli latere. est autem & ambitus rectilineæ figuræ reliquo latere minor; quoniam & minor est circuli ambitu. quare figura rectilinea minor est triangulo e: quod est absurdum.

Sit deinde, si fieri potest, circulus minor triangulo e: & circumscribatur quadratum: circũferentiisq; bifariam sectis, per ea puncta contingentes lineæ ducãtur. erit angulus o a r rectus. & idcirco linea o r maior, quàm r m; quòd r m ipsi r a sit æqualis. triangulum igitur r o p maius est, quàm dimidium figuræ o f a m. itaque sumantur portiones, ipsi p f a similes; quæ quidem minores sint eo, quo triangulum e excedit circulum a b c d. erit figura circumscripta adhuc triangulo e minor: quod item est absurdum, cum sit maior: nam ipsa quidem n a æqualis est trianguli catheto: ambitus uero maior est basi eiusdem. ex quibus sequitur circulum triangulo e æqualem esse.

PROPOSITIO II.

Circulus ad quadratũ diametri eam proportionẽ habet, quam XI ad XIIII.

SIT circulus, cuius diameter a b: & circumscribatur quadratũ c g: & ipsius c d dupla sit d e: sit autem e f, septima eiusdẽ c d. Quo-

(*Archimedis opera*, edited by Commandino, 1558) — see *Mathematics emerging*, §1.2.3

A circle is equal to a right-angled triangle with height equal to the circumference of the circle and base equal to the radius.

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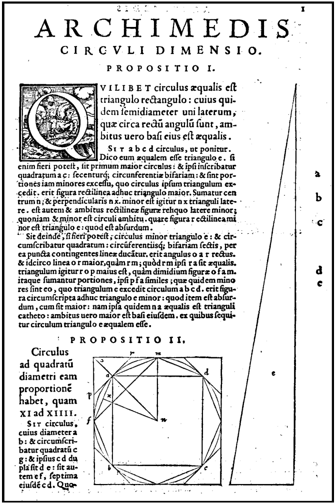

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Proof by exhaustion and double contradiction

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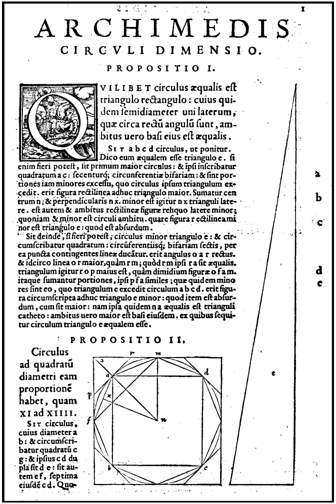

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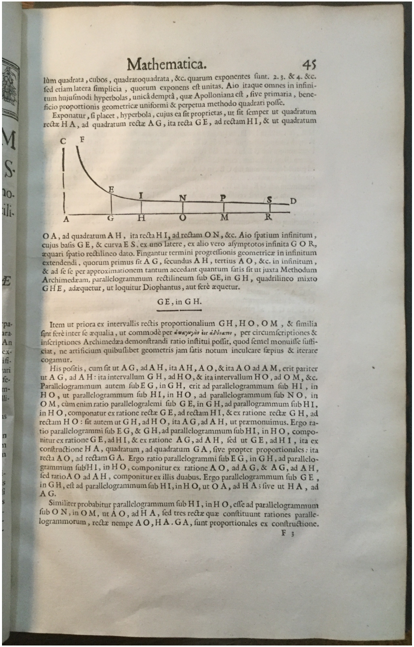
Later: the ratio of the circumference to the diameter is greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$.

(*Archimedis opera*, edited by Commandino, 1558) — see *Mathematics emerging*, §1.2.3

# Fermat's quadrature of a hyperbola (c. 1636)



Mathematica. 45

lùm quadrata, cubos, quadratoquadrata, &c. quarum exponentes ſunt. 2. 3. & 4. &c. ſed etiam latera ſimplicia, quorum exponens eſt unitas. Aio itaque omnes in infinitum hujuſmodi hyperbolas, unicâ demptâ, quæ Apolloniana eſt, ſive primaria, beneficio proportionis geometricæ uniformi & perpetua methodo quadrari poſſe.

Exponatur, ſi placet, hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ H A, ad quadratum rectæ A G, ita recta G E, ad rectam H I, & ut quadratum



O A, ad quadratum A H, ita recta H I, ad rectam O N, &c. Aio ſpatium infinitum, cujus baſis G E, & curva E S, ex uno latere, ex alio vero aſymptotos infinita G O R, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis geometricæ in infinitum extendendi, quorum primus ſit A G, ſecundus A H, tertius A O, &c. in infinitum, & ad ſe ſe per approximationem tantum accedant quantum ſatis ſit ut juxta Methodum Archimedæam, parallelogrammum rectilineum ſub GE, in G H, quadrilineo mixto G H E, adæquetur, ut loquitur Diophantus, aut ferè æquetur.

G E, in G H.

Item ut priora ex intervallis rectis proportionalium G H, H O, O M, & ſimilia ſint ferè inter ſe æqualia, ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones & inſcriptiones Archimedæa demonſtrandi ratio inſtitui poſſit, quod ſemel monuiſſe ſufficiat, ne artificium quibuſlibet geometris jam ſatis notum inculcare ſæpius & iterare cogamur.

His poſitis, cum ſit ut A G, ad A H, ita A H, A O, & ita A O ad A M, erit pariter ut A G, ad A H: ita intervallum G H, ad H O, & ita intervallum H O, ad O M, &c. Parallelogrammum autem ſub E G, in G H, erit ad parallelogrammum ſub H I, in H O, ut parallelogrammum ſub H I, in H O, ad parallelogrammum ſub N O, in O M, cùm enim ratio parallelogralemi ſub G E, in G H, ad parallogrammum ſub H I, in H O, componatur ex ratione rectæ G E, ad rectam H I, & ex ratione rectæ G H, ad rectam H O: ſit autem ut G H, ad H O, ita A G, ad A H, ut præmonuimus. Ergo ratio parallelogrammi ſub E G, & G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione G E, ad H I, & ex ratione A G, ad A H, ſed ut G E, ad H I, ita ex conſtructione H A, quadratum, ad quadratum G A, ſive propter proportionales: ita recta A O, ad rectam G A. Ergo ratio parallelogrammi ſub E G, in G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione A O, ad A G, & A G, ad A H, ſed ratio A O ad A H, componitur ex illis duabus. Ergo parallelogrammum ſub G E, in G H, eſt ad parallelogrammum ſub H I, in H O, ut O A, ad H A; ſive ut H A, ad A G.

Similiter probabitur parallelogrammum ſub H I, in H O, eſſe ad parallelogrammum ſub O N, in O M, ut A O, ad H A, ſed tres rectæ quæ conſtituunt rationes parallelogrammorum, rectæ nempe A O, H A . G A, ſunt proportionales ex conſtructione.

F 3

Worked out c. 1636, but only published posthumously in *Varia opera mathematica*, 1679.

In modern terms, this is the curve described by $y = \frac{1}{x^2}$.

See *Mathematics emerging*, §3.2.1.

# The rectangular (or 'Apollonian') hyperbola

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- Empirical observation that if $A(x)$ is the area under the hyperbola from 1 to $x$, then $A(\alpha\beta) = A(\alpha) + A(\beta)$ (cf. logarithms)

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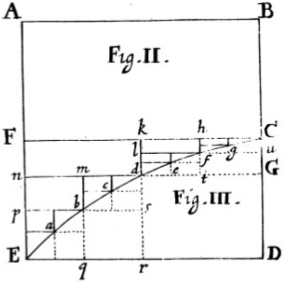
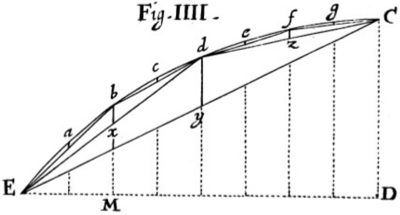
- Problem solved in early 1650s by William Brouncker; published in 1668 in volume 3 of *Philosophical Transactions of the Royal Society*

# Brouncker's quadrature of the hyperbola (1668)




To put this into modern terms, take $A$ as the origin, and $AB$, $AE$ as the $x$- and $y$-axes, respectively. Then the diagram represents the area under $\dfrac{1}{1+x}$ from $x = 0$ to $x = 1$.

(See *Mathematics emerging*, §3.2.2.)

# Brouncker's article of 1668



**PHILOSOPHICAL TRANSACTIONS.**

Monday, *April* 13. 1668

The Contents.

*The Squaring of the* Hyperbola *by an infinite ſeries of* Rational *Numbers, together with its Demonſtration, by the Right Honourable the Lord Viſcount* Brouncker. *An Extract of a Letter ſent from* Danzick, *touching ſome Chymical, Medicinal and Anatomical particulars. Two Letters, written by Dr.* John Wallis *to the* Publiſher; *One, concerning the Variety of the Annual High-Tides in reſpect to ſeveral places: the other, concerning ſome Miſtakes of a Book entituled SPECIMINA MATHEMATICA* Franciſci Dulaurens, *eſpecially touching a certain* Probleme, *affirm'd to have been propoſed by Dr.* Wallis *to the Mathematicians of all* Europe, *for a ſolution. An Account of ſome Obſervations concerning the* true Time *of the Tydes, by Mr.* Hen. Philips. *An Account of three Books: I. W.SENGWERDIUS PH.D.* de Tarantula. *II. REGNERI* de *GRAEF M.D.* Epiſtola de nonnullis circa Partes *Genitales* Inventis Novis. *III. JOHANNIS* van *HORNE M.D.* Obſervationum ſuarum circa Partes *Genitales* in utroque ſexu, *PRODROMUS.*

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WHat the Acute Dr. *John Wallis* had intimated, ſome years ſince, in the Dedication of his Anſwer to M. *Meibomius de proportionibus, vid.* That the World one day would learn from the Noble Lord *Brounker*, the *Quadrature* of the *Hyperbole*; the Ingenious Reader may ſee performed in the ſubjoyned operation, which its Excellent Author was now pleaſed to communicate, as followeth in his own words;

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*My Method for Squaring the* Hyperbola *is this:*

Let AB be one *Asymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Letter x every where stands for Multiplication.

Supposing the Reader knows, that EA. $\alpha\zeta$. KH. $\beta\nu$. d$\theta$. $\gamma\kappa$. $\delta\lambda$. $\epsilon\mu$. CB. &c. are in an *Harmonic* series, or a *series reciproca primanorum seu arithmetice proportionalium* (otherwise he is referr'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Wallisij:*)

I say
$$\left.\begin{aligned} ABCdEA &= \frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\frac{1}{7\times8}+\frac{1}{9\times10}\ \&c. \\ EdCDE &= \frac{1}{2\times3}+\frac{1}{4\times5}+\frac{1}{6\times7}+\frac{1}{8\times9}+\frac{1}{10\times11}\ \&c. \\ EdCyE &= \frac{1}{2\times3\times4}+\frac{1}{4\times5\times6}+\frac{1}{6\times7\times8}+\frac{1}{8\times9\times10}\ \&c. \end{aligned}\right\}\ \textit{in infinitum.}$$

| For (in Fig. 2, & 3) the Parallelog. | | And (in Fig. 4.) the Triangl. | *Note.* |
|---|---|---|---|
| $CA = \frac{1}{1\times2}$ | | $EdC = \frac{1}{2\times3\times4} = \frac{\square dD - \square dF}{2}$ | |
| $dD = \frac{1}{2\times3}$ | $dF = \frac{1}{3\times4}$ | $Ebd = \frac{1}{4\times5\times6} = \frac{\square br - \square bn}{2}$ | $CA = dD + dF$ |
| $br = \frac{1}{4\times5}$ | $bn = \frac{1}{5\times6}$ | $dfC = \frac{1}{6\times7\times8} = \frac{\square fG - \square fk}{2}$ | $dD = br + bn$ |
| $fG = \frac{1}{6\times7}$ | $fk = \frac{1}{7\times8}$ | $Eab = \frac{1}{8\times9\times10} = \frac{\square aq - \square ap}{2}$ | $dF = fG + fk$ |
| $aq = \frac{1}{8\times9}$ | $ap = \frac{1}{9\times10}$ | $bcd = \frac{1}{10\times11\times12} = \frac{\square cs - \square cm}{2}$ | $br = aq + ap$ |
| $cs = \frac{1}{10\times11}$ | $cm = \frac{1}{11\times12}$ | $def = \frac{1}{12\times13\times14} = \frac{\square et - \square el}{2}$ | $bn = cs + cm$ |
| $et = \frac{1}{12\times13}$ | $el = \frac{1}{13\times14}$ | $fgC = \frac{1}{14\times15\times16} = \frac{\square gu - \square gh}{2}$ | $fG = et + el$ |
| $gu = \frac{1}{14\times15}$ | $gh = \frac{1}{15\times16}$ | &c. | $fk = gu + gh$ |
| &c. | &c. | | &c. |