BO1 History of Mathematics
Lecture IV
The beginnings of calculus, continued
Part 2: Indivisibles and infinitesimals

MT 2020 Week 2

New methods: indivisibles and infinitesimals

Indivisibles: geometric objects making up a higher-dimensional object (e.g., points $\to$ line, lines $\to$ plane)

Infinitesimal: arbitrarily small but nonzero quantity

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Infinitesimal: arbitrarily small but nonzero quantity

But distinction often blurred

During the 17th century, both concepts saw much use — despite the fact that they appeared to contradict Euclidean principles

Indivisibles

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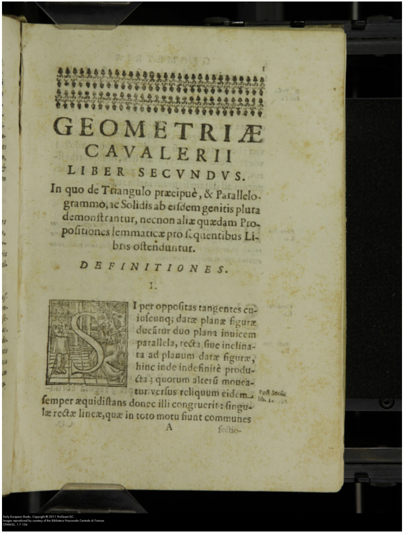
Developed by John Wallis (1616–1703) and others.

Cavalieri's *Geometria*



I

GEOMETRIÆ
CAVALERII
LIBER SECVNDVS.

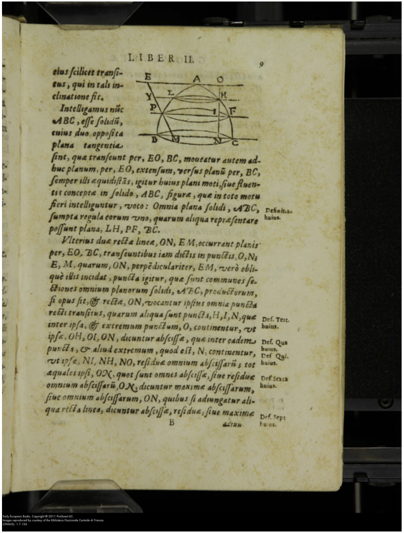
In quo de Triangulo præcipuè, & Parallelogrammo, ac Solidis ab eisdem genitis plura demonstrantur, necnon aliæ quædam Propositiones lemmaticæ pro sequentibus Libris ostenduntur.

DEFINITIONES.

I.

SI per oppositas tangentes cuiuscunq; datæ planæ figuræ ducãtur duo plana inuicem parallela, recta, siue inclinata ad planum datæ figuræ, hinc inde indefinitè producta; quorum alterũ moueatur versus reliquum eidem semper æquidistans donec illi congruerit: singulæ rectæ lineæ, quæ in toto motu fiunt communes Post Secũ. lib. 1.

A sectio-



LIBER II. 9

eius scilicet transitus, qui in tali inclinatione fit.

Intelligamus nũc ABC, esse solidũ, cuius duo opposita plana tangentia sint, quæ transeunt per, EO, BC, moueatur autem adhuc planum, per, EO, extensum, versus planũ per, BC, semper illi æquidistãs, igitur huius plani moti, siue fluentis conceptæ in solido, ABC, figuræ, quæ in toto motu fieri intelliguntur, voco: Omnia plana solidi, ABC, sumpta regula eorum vno, quarum aliqua repræsentare possunt plana, LH, PF, BC. Definit. 2. huius.

Vlterius duæ rectæ lineæ, ON, EM, occurrant planis per, EO, BC, transeuntibus iam dictis in punctis, O, N; E, M, quarum, ON, perpẽdiculariter, EM, verò obliquè illis incidat, puncta igitur, quæ sunt communes sectiones omnium planorum solidi, ABC, productorum, si opus sit, & rectæ, ON, vocantur ipsius omnia puncta recti transitus, quarum aliqua sunt puncta, H, I, N, quæ inter ipsa, & extremum punctum, O, continentur, vt ipsæ, OH, OI, ON, dicuntur abscissæ, quæ inter eadem puncta, & aliud extremum, quod est, N, continentur, vt ipsæ, NI, NH, NO, residuæ omnium abscissarũ; tot æquales ipsi, ON, quot sunt omnes abscissæ, siue residuæ omnium abscissarũ, ON, dicuntur maximæ abscissarum, siue omnium abscissarum, ON, quibus si adiungatur aliqua recta linea, dicuntur abscissæ, residuæ, siue maximæ Def. Tert. huius. Def. Qua huius. Def Qui. huius. Def. Sexta huius. Def. Sept huius.

B *adiun-*

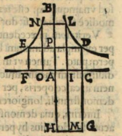
Torricelli's hyperbolic solid (*Opera geometrica*, 1644)

Lemma V.

Cuiuscunque cylindri ghil *intra solidum acutum descripti (vt in præcedenti figura) superficies sine basibus æqualis est circulo cuius semidiameter sit linea* df. *nempe semiaxis, siue semilatus versum ipsius hyperbolæ. Hoc enim in ipso progressu præcedentis lemmatis demonstratum est.*

Theorema.

Solidum acutum hyperbolicum infinitè longum, sectum plano ad axem erecto, vnà cum cylindro suæ basis, æquale est cylindro cuidam recto, cuius basis diameter sit latus versum, siue axis hyperbolæ, altitudo verò sit æqualis semidiametro basis ipsius acuti solidi.



Esto hyperbola cuius asymptoti *ab*, *ac* angulum rectum contineant; sumptoq; in hyperbola quolibet puncto *d*, ducatur *dc* æquidistans ipsi *ab*, & *dp* æquidistans *ac*. Tū conuertatur vniuersa figura circa axē *ab*. ità vt fiat solidum acutum byperbolicum *ebd*, vna cum cylindro suæ basis *fedc*. Poducatur *ba* in *h*. ita vt *ah*. æqualis sit integro axi, siue lateri verso hyperbolæ. Et circa diametrum *ah* intelligatur circulus erectus ad asymptoton *ac*: & super basi *ah* concipiatur cylindrus rectus *acgh*, cuius altitudo sit *ac*, nempe semidiameter basis acuti solidi. Dico solidum vniuersum *febdc*, quanquam sine fine longum, æquale tamen esse cylindro *acgh*.

Accipiatur in recta *ac* quodlibet punctum *i*, & per *i* intelligatur ducta superficies cylindrica *onli* in solido acuto

Torricelli's hyperbolic solid (*Opera geometrica*, 1644)

Problema Secundum. 115

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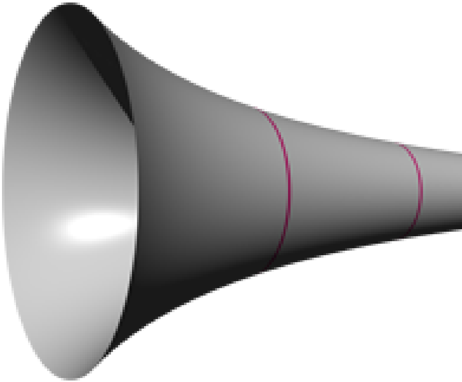
P 2 com-



(See *Mathematics emerging*, §3.3.1.)

John Wallis (1616–1703)

Studied at Emmanuel College, Cambridge (BA 1637, MA 1640)

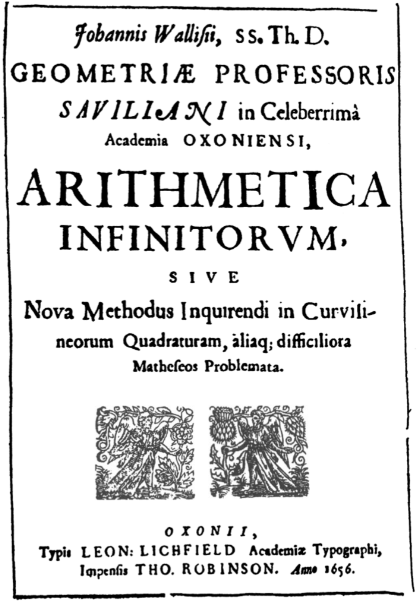
1643–1649: scribe for Westminster Assembly

1644–1645: Fellow of Queens' College, Cambridge

1643–1689: cryptographer to Parliament, then to the Crown

1649–1703: Savilian Professor of Geometry in Oxford

Arithmetica infinitorum



Johannis Wallisii, SS. Th. D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimà
Academia OXONIENSI,

ARITHMETICA
INFINITORVM,

SIVE

Nova Methodus Inquirendi in Curvilineorum Quadraturam, àliaq; difficiliora Matheſeos Problemata.

OXONII,
Typis LEON: LICHFIELD Academiæ Typographi,
Impenſis THO. ROBINSON. Anno 1656.

John Wallis,
Arithmetica infinitorum
(*The arithmetic of infinitesimals*)
Oxford, 1656

Translation by
Jacqueline A. Stedall
Springer, 2004

Arithmetica infinitorum

- Arithmetical methods rather than geometrical

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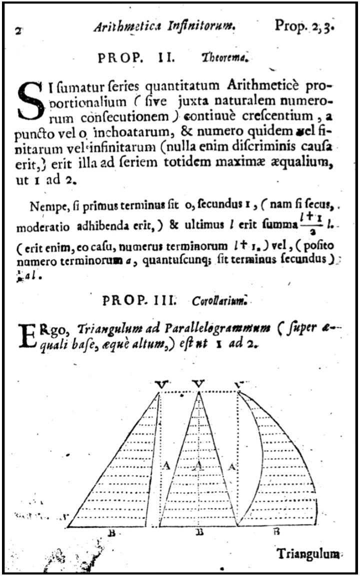
- Investigation of sums of sequences of powers (or ratios of these to a known fixed quantity) — usually decreasing
- Fixed an endpoint, dividing interval into infinite number of arbitrarily small subintervals — these are the 'infinitesimals' of Wallis' title

Wallis and indivisibles



2 *Arithmetica Infinitorum.* Prop. 2, 3.

PROP. II. *Theorema.*

SI ſumatur ſeries quantitatum Arithmeticè proportionalium (ſive juxta naturalem numerorum conſecutionem) continuè creſcentium, a puncto vel o inchoatarum, & numero quidem vel finitarum vel infinitarum (nulla enim diſcriminis cauſa erit,) erit illa ad ſeriem totidem maximæ æqualium, ut 1 ad 2.

Nempe, ſi primus terminus ſit o, ſecundus 1, (nam ſi ſecus, moderatio adhibenda erit,) & ultimus l erit ſumma $\frac{l+1}{2}l$. (erit enim, eo caſu, numerus terminorum $l+1$.) vel, (poſito numero terminorum a, quantuſcunq; ſit terminus ſecundus) $\frac{1}{2}al$.

PROP. III. *Corollarium.*

ERgo, *Triangulum ad Parallelogrammum (ſuper æquali baſe, æquè altum,) eſt ut* 1 *ad* 2.

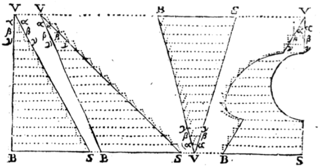
Triangulum

For the triangle . . . consists of an infinite number of parallel lines in arithmetic proportion . . .

(See *Mathematics emerging*, §2.4.2.)

Wallis and indivisibles?

ralis initium. Quamvis enim Sectorum illorum numero infinitorum aggregatum, ipſi figuræ lineis recta & Spirali terminatæ, (juxta methodum Indiviſibilium) æquale ponatur; non tamen illud de omnium Arcubus cum ipſa Spirali (propriè dicta) comparatis obtinebit. Tantundem enim eſſet, acſi quis, dum infinita numero parallelogramma triangulo inſcripta (aut etiam circumſcripta) toti triangulo VBS æqualia videat, inde



concluderet eorum omnium latera rectæ VS adjacentia (rectæ VB parallela) ipſi VS ſimul æqualia eſſe, vel quæ rectæ VB adjacent (ipſi VS parallela) æqualia ſimul eſſe toti VB. (Quod ſiquando verum eſſe contingat, puta in triangulo iſoſceli, non tamen id univerſaliter concludendum erit.) Atq; hoc quidem eo potius admonendum duxi, quod viderim etiam viros doctos nonnunquam ſpecioſa ejuſmodi veriſimilitudine in lapſum proclives eſſe. Cur autem omiſſa Spirali genuina, ſpuriam hanc peripheriæ compararerim; cauſa eſt, quod huic poſſim, non autem illi, æqualem peripheriam aſſignare.

PROP. XIV. *Corollarium.*

ET propterea etiam *ſegmenta ſpiralis, a principio ſpiralis exorſa, ſunt ad rectas conterminas, ſicut Parabolæ Diametri interceptæ, ad ordinatim-applicatas.*

For it amounts to the same thing as if, when an infinite number of parallelograms are inscribed in (or circumscribed about) a triangle, it seems that they equal to complete triangle ...

(See *Mathematics emerging*, §2.4.2.)

Sums of powers

Wallis' method depended upon the summation rule

$$\sum_{a=0}^{A} a^n \approx \frac{A^{n+1}}{n+1}$$

This was known to Fermat, Roberval, and Cavalieri in the 1630s for positive integers n

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Simple ‘integrals’

Using the summation rule we can ‘integrate’

$$x^2, \quad x^3, \quad \ldots, \quad x^{1/3}, \quad \ldots, \quad x^{-4}, \quad \ldots$$

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$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad \dots$$

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and

$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad \ldots$$

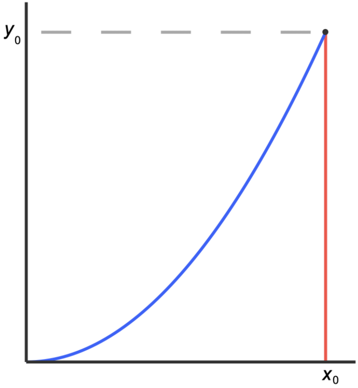
but what about

$$(1-x^2)^{1/2} \quad \text{[for a circle]}$$

or

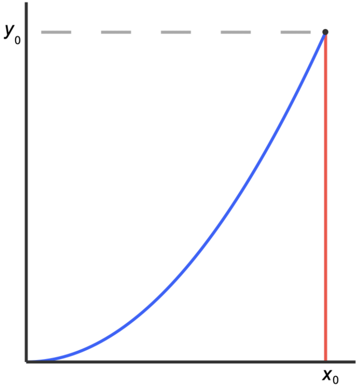
$$(1+x)^{-1} \quad \text{[for a hyperbola] ?}$$

Wallis and the quadrature of a parabola



Wallis sought the area under the parabola $y = x^2$ between $x = 0$ and $x = x_0$

Wallis and the quadrature of a parabola



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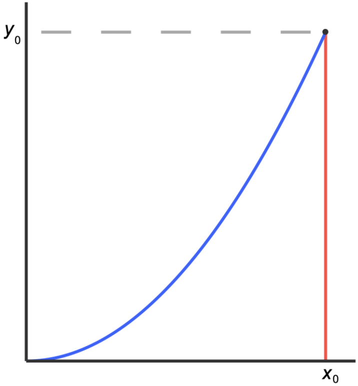
He used the language of ratio, hence sought to calculate the ratio of the area A under the curve to that of the corresponding rectangle $(x_0 y_0)$, which we may think of as the fraction $\frac{A}{x_0 y_0}$

Wallis and the quadrature of a parabola



Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?)

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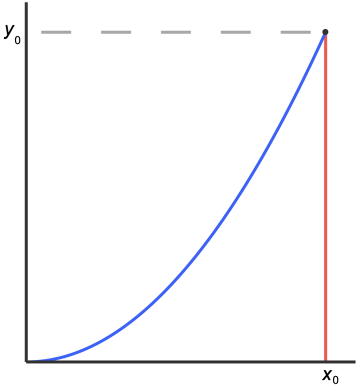
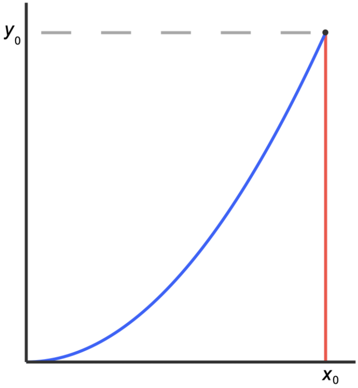


Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?), so

- A is the sum of the values of x^2 as x ranges from 0 to x_0
- $x_0 y_0$ is the sum of as many copies of x_0^2 (?)

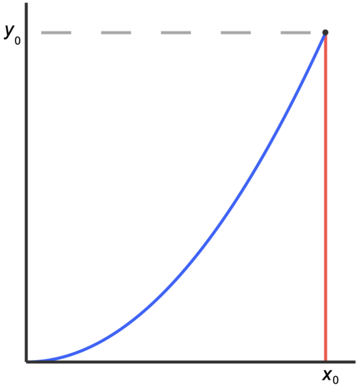
Wallis and the quadrature of a parabola



Break $(0, x_0)$ into n subintervals, suppose that x only takes the values at the endpoints of these, and consider the ratio

$$R = \frac{0^2 + 1^2 + 2^2 + \cdots + n^2}{n^2 + n^2 + n^2 + \cdots + n^2}$$

Wallis and the quadrature of a parabola



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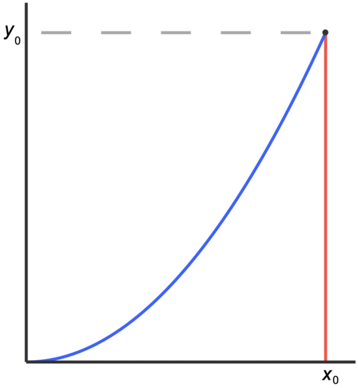
As we make n larger, this ratio will become a closer approximation to $\frac{A}{x_0 y_0}$

Wallis and the quadrature of a parabola



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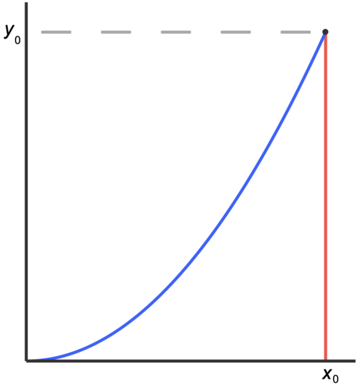
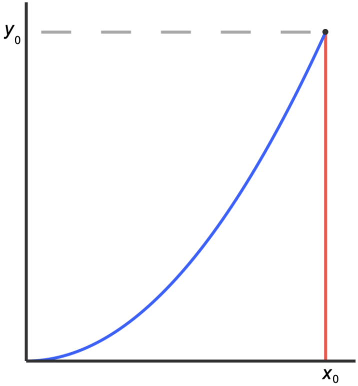
[Note that we are deliberately avoiding the terminology of limits, and that some x_0^2s have been cancelled, thanks to the use of ratios]

Wallis and the quadrature of a parabola



Wallis investigated the cases of small n

Wallis and the quadrature of a parabola



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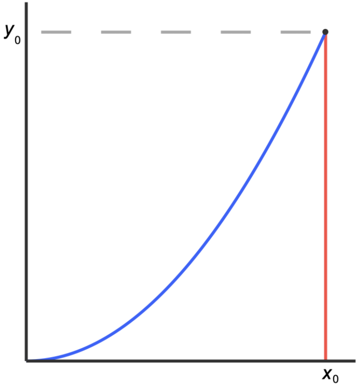
For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2}$$

Wallis and the quadrature of a parabola



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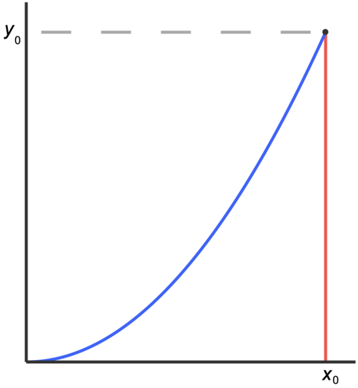
For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2}$$

Wallis and the quadrature of a parabola

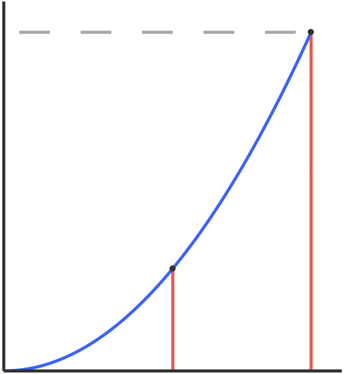


Wallis investigated the cases of small n

For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

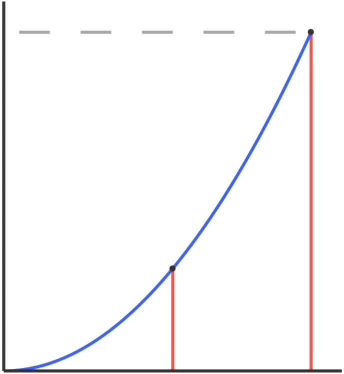
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2}$$

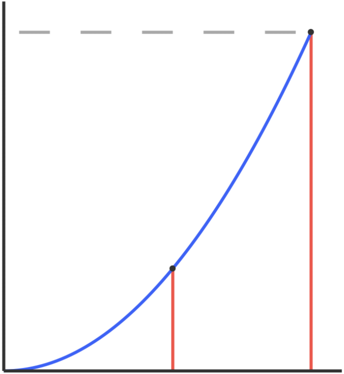
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12}$$

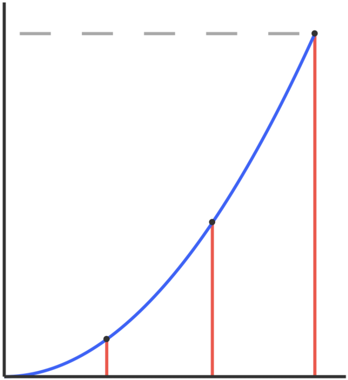
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

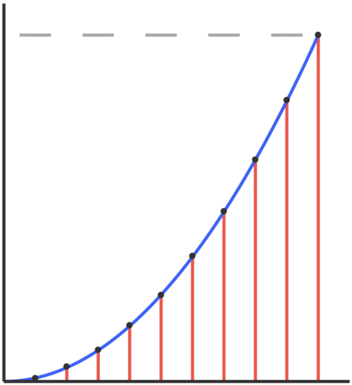
Wallis and the quadrature of a parabola



For $n = 3$ (three red lines),

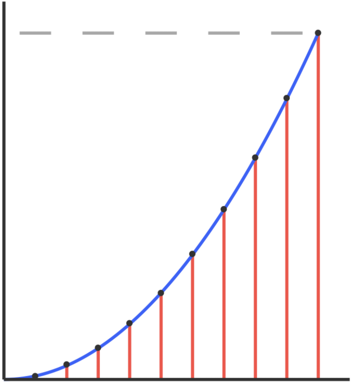
$$R = \frac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2}$$
$$= \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$

Wallis and the quadrature of a parabola



So as n increases $\frac{A}{x_0 y_0}$ approaches $\frac{1}{3}$, hence $A = \frac{1}{3}x_0^3$, as we'd expect

Wallis and the quadrature of a parabola



So as n increases $\frac{A}{x_0 y_0}$ approaches $\frac{1}{3}$, hence $A = \frac{1}{3}x_0^3$, as we'd expect

Wallis called this method of spotting and extending a pattern 'induction' — it was criticised at the time (for example, by Pascal)