BO1 History of Mathematics
Lecture VIII
Establishing rigorous thinking in analysis
Part 1: Early rigour

MT 2020 Week 4

Summary

Part 1

- French institutions
- Fourier series
- Early-19th-century rigour

Part 2

- Limits, continuity, differentiability
- Mathematics of small quantities
- The baton passes from France to Germany

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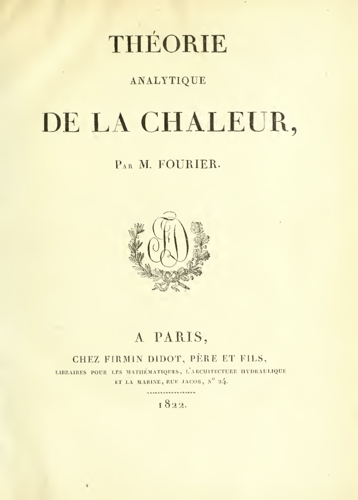
- and a new focus on rigour

Fourier series


THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.

A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

Joseph Fourier, *Analytic theory of heat*, 1822

See: Bernard Maurey, 'Fourier, one man, several lives', *European Mathematical Society Newsletter*, no. 113 (Sept 2019), 8–20

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BUT it led to profound results

Establishing rigour

The development of ‘rigour’:

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The development of 'rigour':

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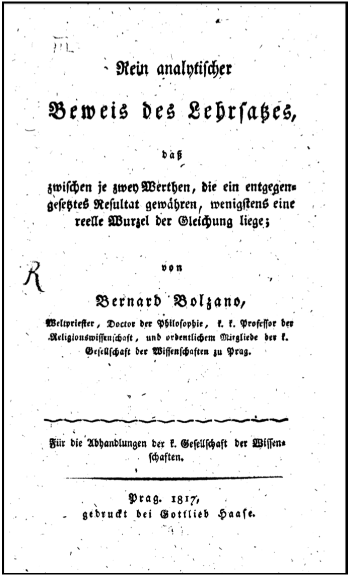
- ϵ, δ notation

Cauchy sequences: Bolzano (1817)



Rein analytischer

Beweis des Lehrsatzes,

daß

zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;

von

Bernard Bolzano,

Weltpriester, Doctor der Philosophie, k. k. Professor der Religionswissenschaft, und ordentlichem Mitgliede der k. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der k. Gesellschaft der Wissenschaften.

Prag. 1817,
gedruckt bei Gottlieb Haase.

Bernard Bolzano, *Purely analytic proof of the theorem that between any two values which give opposite values lies at least one real root of the equation*, 1817

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Began with a discussion of previous proofs of the Intermediate Value Theorem (by "mathematicians of great repute")

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Cauchy sequences: Bolzano (1817)

If a series of quantities has the property that the difference between its n-th term and every later one remains smaller than any given quantity . . . then there is always a certain constant quantity . . . which the terms of this series approach.

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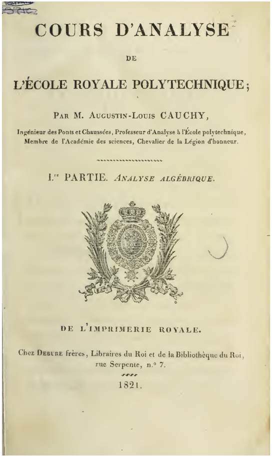
> *If a series of quantities has the property that the difference between its n-th term and every later one remains smaller than any given quantity ... then there is always a certain constant quantity ... which the terms of this series approach.*

Proof: The hypothesis that there exists a quantity X which the terms of this series approach ... contains nothing impossible ...

(See: *Mathematics emerging*, §16.1.1; for a full translation, see: S. B. Russ, A translation of Bolzano's paper on the intermediate value theorem, *Historia Mathematica* **7**(2) (1980), 156–185)

Cauchy's *Cours d'analyse*

Augustin-Louis Cauchy, *Cours d'analyse de l'École royale polytechnique* (1821)



COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. *ANALYSE ALGÉBRIQUE.*

DE L'IMPRIMERIE ROYALE.

Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821.

(Annotated translation by Robert E. Bradley and C. Edward Sandifer, Springer, 2009)

Cauchy sequences: Cauchy (1821)

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0,\ u_1,\ u_2,\ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit s: in other words, it is necessary and sufficient that for infinitely large values of the number n, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit s, and consequently from each other, by infinitely small quantities.*

(See: *Mathematics emerging*, §16.1.2.)

More from Cauchy (1821)

Further results from Cauchy's *Cours d'analyse*:

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- ratio test;

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Further results from Cauchy's *Cours d'analyse*:

- ratio test;
- root test;
- alternating series test (proof uses Cauchy sequences);
- and many more.

(See *Mathematics emerging*, §16.1.2)

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- in Abel's proof of the general binomial theorem (1826)

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BUT the convergence of Cauchy sequences themselves remained unproved