BO1 History of Mathematics
Lecture VIII
Establishing rigorous thinking in analysis
Part 2: Further rigour

MT 2020 Week 4

Continuity

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[Question: dependence? plagiarism? or a common source?]

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D'Alembert (1751):	'one may approach a limit as closely as one wishes … but never surpass it'; example: polygons and circle; he assumed that $\lim AB = \lim A \times \lim B$; a dictionary definition only — no theory

Limits: a later definition

Cauchy, *Cours d'analyse* (1821), p. 4:

> *When the values successively given to a variable approach indefinitely to a fixed value, so as to finish by differing from it by as little as one would wish, the latter is called the limit of all the others.*

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BUT still no formal definition of

- 'as small as one wishes',
- 'as closely as one wishes', ...

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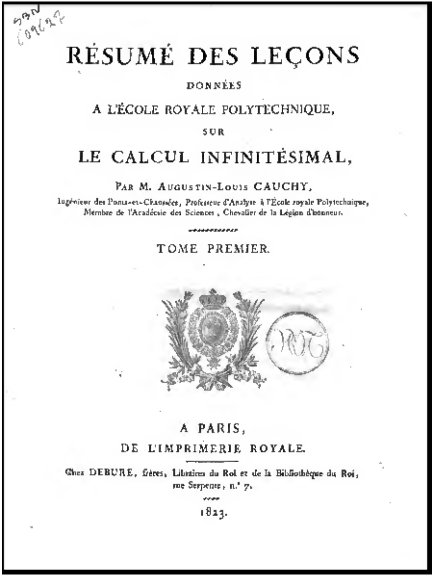
Ampère (1806) struggled with the meaning of

$$\frac{f(x+h) - f(x)}{h}$$

— why isn’t it just zero or infinite?

Differentiability: Cauchy's *Résumé*



RÉSUMÉ DES LEÇONS
DONNÉES
A L'ÉCOLE ROYALE POLYTECHNIQUE,
SUR
LE CALCUL INFINITÉSIMAL,
PAR M. AUGUSTIN-LOUIS CAUCHY,
Ingénieur des Ponts-et-Chaussées, Professeur d'Analyse à l'École royale Polytechnique, Membre de l'Académie des Sciences, Chevalier de la Légion d'honneur.

TOME PREMIER.

A PARIS,
DE L'IMPRIMERIE ROYALE.
Chez DEBURE, frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.
1823.

Cauchy, *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, 1823

(Translation by Dennis M. Cates, Fairview Academic Press, 2012)

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but no concerns about existence in general

(See: *Mathematics emerging*, §14.1.4.)

Arbitrarily small intervals

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Proof: Divide the interval $[a, b]$ into n subintervals, taking n as large as necessary ...

Unconvincing to modern eyes, but a useful technique.

(See: *Mathematics emerging*, §11.2.3.)

IVT revisited

Cauchy, *Cours d'analyse* (1821), Note III, p. 460 (On the numerical solution of equations):

Theorem: Let f be a real function of the variable x, which remains continuous with respect to this variable between the limits $x = x_0$, $x = X$. If the two quantities $f(x_0)$, $f(X)$ are of opposite signs, the equation $f(x) = 0$ will be satisfied by one or more real values of x contained between x_0 and X.

(See: *Mathematics emerging*, §11.2.6.)

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But it also provides a much more convincing proof of the Intermediate Value Theorem than that appearing earlier in Cauchy's text (*Cours d'analyse*, Ch. II, Theorem 4: p. 44).

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Suppose that in the interval $[x_0, X]$ we have $A < f'(x) < B$. Then we also have

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Proof: Choose two quantities $\epsilon,\ \delta, \ldots$ such that for $i < \delta$

$$f'(x) - \epsilon < \frac{f(x+i) - f(x)}{i} < f'(x) + \epsilon$$

etc.

(See: *Mathematics emerging*, §14.1.5.)

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So not every function can be expanded into a Taylor series, and it appears to be possible to conceive of functions to which the calculus is not immediately or naturally applicable ...

Modern rigour in analysis

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BUT we have no direct sources, only lecture notes or books by his pupils and followers

From France to Germany

By the later 19th century the mathematical centre of gravity in Europe had moved from the Parisian Écoles to the German universities:

Göttingen (est. 1734):	Gauss, Dirichlet, [Dedekind], Riemann, Klein, Hilbert, ...
Berlin (est. 1810):	Crelle (editor), Dirichlet, Eisenstein, Kummer, [Jacobi], Kronecker, Weierstrass, ...

with a focus on both research and teaching.