# BO1 History of Mathematics
# Lecture IX
# Classical algebra: equation solving 1800BC – AD1800
# Part 1: Quadratics, cubics, and quartics

MT 2020 Week 5

# Summary

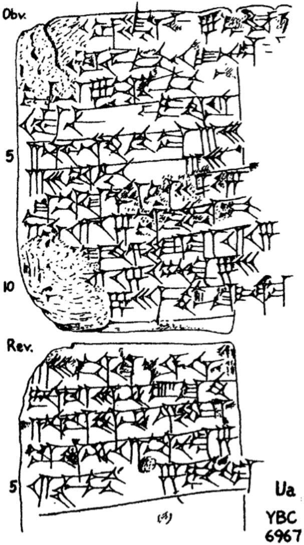
Part 1

- Early quadratic equations
- Cubic and quartic equations
- Further 16th-century developments

Part 2

- 17th century ideas
- 18th century ideas
- Looking back

# Completing the square, c. 1800 BC



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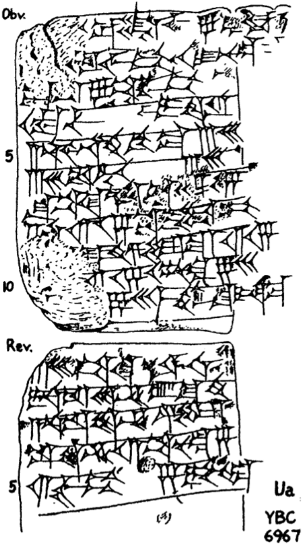


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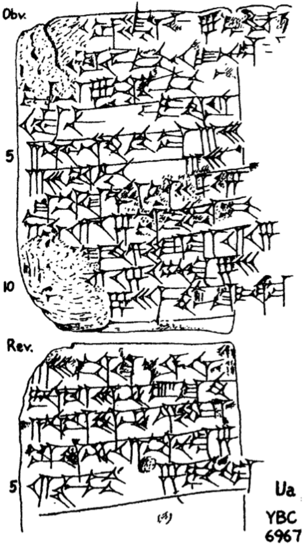


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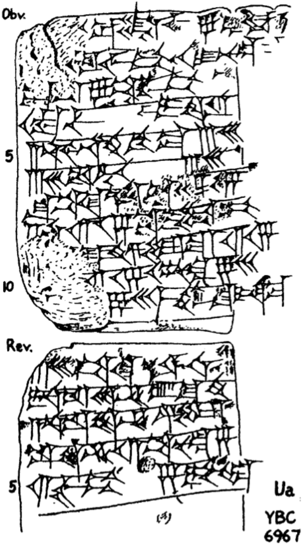


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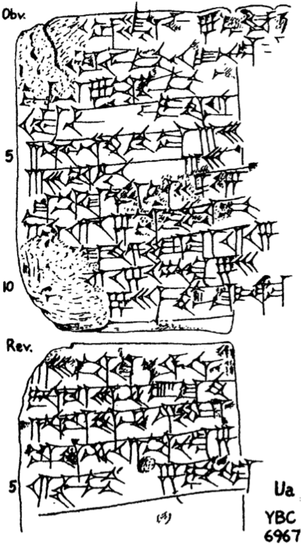


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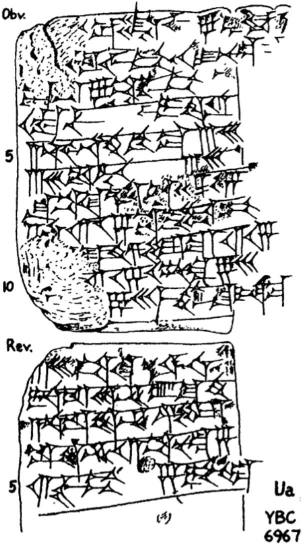


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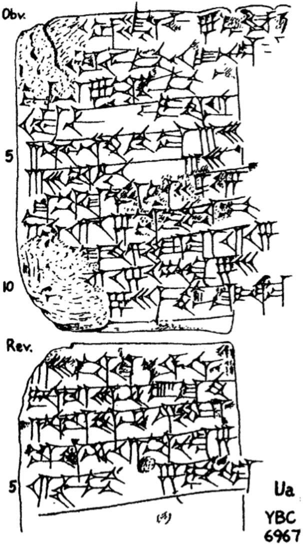


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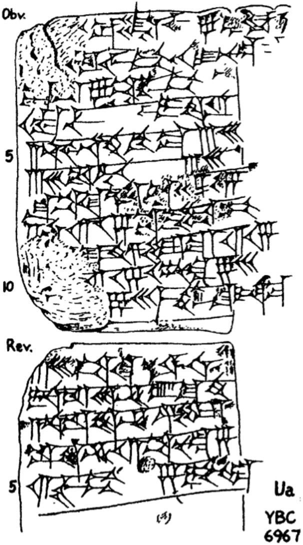


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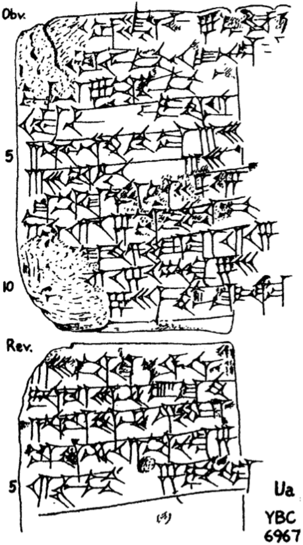


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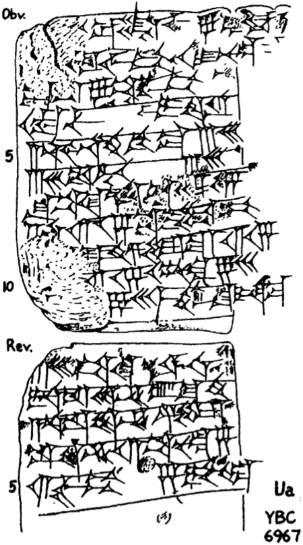


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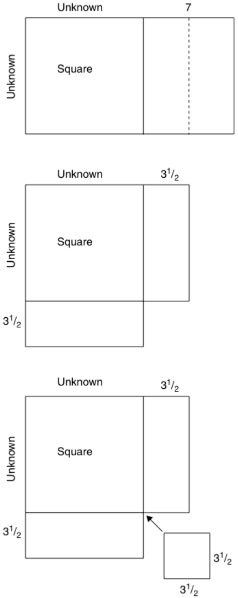


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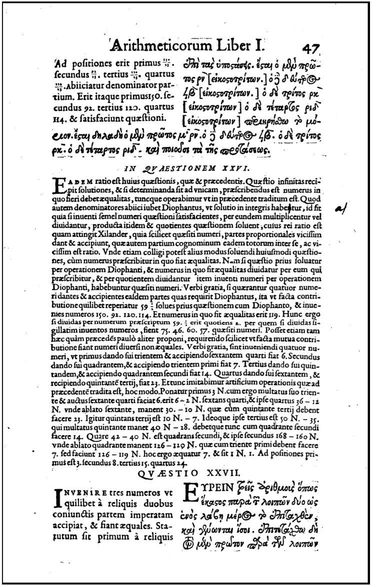
- Solution recipe derived from geometrical insight
- Not (explicitly) a general solution — but reader ought to be able to adapt the method
- Is this algebra? Geometrical algebra?

# Diophantus of Alexandria (3rd century AD)





Ad poſitiones erit primus $\frac{150}{23}$. ſecundus $\frac{92}{23}$. tertius $\frac{120}{23}$. quartus $\frac{114}{23}$. Abiiciatur denominator partium. Erit itaque primus 150. ſecundus 92. tertius 120. quartus 114. & ſatisfaciunt quæſtioni.

ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν πρῶτος ρν [εἰκοστοτρίτων.] ὁ δὲ δεύτερος ϙβ [εἰκοστοτρίτων.] ὁ δὲ τρίτος ρκ [εἰκοστοτρίτων] ὁ δὲ τέταρτος ριδ [εἰκοστοτρίτων] περιῃρήσθω τὸ μόριον. ἔσται δηλαδὴ ὁ μὲν πρῶτος μο ρν. ὁ δὲ δεύτερος ϙβ. ὁ δὲ τρίτος ρκ. ὁ δὲ τέταρτος ριδ. καὶ ποιοῦσι τὰ τῆς προτάσεως.

*IN QVÆSTIONEM XXVI.*

EADEM ratio eſt huius quæſtionis, quæ & præcedentis. Quæſtio infinitas recipit ſolutiones, & ſi determinanda ſit ad vnicam, præſcribendus eſt numerus in quo fieri debet æqualitas, tuncque operabimur vt in præcedente traditum eſt. Quod autem denominatores abiici iubet Diophantus, vt ſolutio in integris habeatur, id fit quia ſi inuenti ſemel numeri quæſtioni ſatisfacientes, per eundem multiplicentur vel diuidantur, producta itidem & quotientes quæſtionem ſoluent, cuius rei ratio eſt quam attingit Xilander, quia ſcilicet quæſiti numeri, partes proportionales viciſſim dant & accipiunt, quæ autem partium cognominum eadem totorum inter ſe, ac viciſſim eſt ratio. Vnde etiam colligi poteſt alius modus ſoluendi huiuſmodi quæſtiones, cùm numerus præſcribitur in quo fiat æqualitas. Nam ſi quæſtio prius ſoluatur per operationem Diophanti, & numerus in quo fit æqualitas diuidatur per eum qui præſcribitur, & per quotientem diuidantur item inuenti numeri per operationem Diophanti, habebuntur quæſiti numeri. Verbi gratia, ſi quærantur quatuor numeri dantes & accipientes eaſdem partes quas requirit Diophantus, ita vt facta contributione quilibet reperiatur $59\frac{1}{2}$ ſolues prius quæſtionem cum Diophanto, & inuenies numeros 150. 92. 120. 114. Et numerus in quo fit æqualitas erit 119. Hunc ergo ſi diuidas per numerum præſcriptum $59.\frac{1}{2}$ erit quotiens 2. per quem ſi diuidas ſigillatim inuentos numeros, fient 75. 46. 60. 57. quæſiti numeri. Poſſet etiam tam hæc quàm præcedẽs paulò aliter proponi, requirendo ſcilicet vt facta mutua contributione fiant numeri diuerſi non æquales. Verbi gratia, ſint inueniendi quatuor numeri, vt primus dando ſui trientem & accipiendo ſextantem quarti fiat 6. Secundus dando ſui quadrantem, & accipiendo trientem primi fiat 7. Tertius dando ſui quintandem, & accipiendo quadrantem ſecundi fiat 14. Quartus dando ſui ſextantem, & recipiendo quintantẽ tertij, fiat 23. Et tunc imitabimur artificium operationis quæ ad præcedentẽ tradita eſt, hoc modo. Ponatur primus 3 N. cum ergo multatus ſuo triente & auctus ſextante quarti faciat 6. erit 6 – 2 N. ſextans quarti, & ipſe quartus 36 – 12 N. vnde ablato ſextante, manent 30. – 10 N. quæ cùm quintante tertij debent facere 23. Igitur quintans tertij eſt 10 N. – 7. Ideoque ipſe tertius eſt 50 N. – 35. qui multatus quintante manet 40 N – 28. debetque tunc cum quadrante ſecundi facere 14. Quare 42 – 40 N. eſt quadrans ſecundi, & ipſe ſecundus 168 – 160 N. vnde ablato quadrante manent 126 – 129 N. quæ cum triente primi debent facere 7. ſed faciunt 126 – 119 N. hoc ergo æquatur 7. & fit 1 N. 1. Ad poſitiones primus eſt 3. ſecundus 8. tertius 15. quartus 24.

QVÆSTIO XXVII.

INVENIRE tres numeros vt quilibet à reliquis duobus coniunctis partem imperatam accipiat, & fiant æquales. Statutum ſit primum à reliquis

ΕΥΡΕΙΝ τρεῖς ἀριθμοὺς ὅπως ἕκαστος παρὰ τῶν λοιπῶν δύο ὡς ἑνὸς λάβῃ μέρος τὸ ἐπιταχθὲν, καὶ γίνωνται ἴσοι. ἐπιτετάχθω δὴ τὸν μὲν πρῶτον παρὰ τῶν λοιπῶν

Problem I.27: Find two numbers such that their sum and product are given numbers

# Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

Noted six cases of equations:

1. Squares are equal to roots ($ax^2 = bx$)
2. Squares are equal to numbers ($ax^2 = c$)
3. Roots are equal to numbers ($bx = c$)
4. Squares and roots are equal to numbers ($ax^2 + bx = c$)
5. Squares and numbers are equal to roots ($ax^2 + c = bx$)
6. Roots and numbers are equal to squares ($bx + c = ax^2$)

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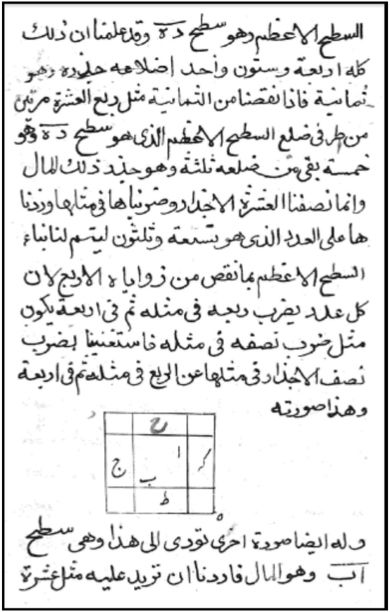


السطح الاعظم وهو سطح د ه وقد علمنا ان ذلك
كله اربعة وستون واحد اضلاعه جذره وهو
ثمانية فاذا نقصنا من الثمانية مثل ربع العشرة مرتين
من طرفي ضلع السطح الاعظم الذي هو سطح د ه فهو
خمسة بقي من ضلعه ثلثة وهو جذر ذلك المال
وانما نصفنا العشرة الاجذار وضربناها في مثلها وزدنا
ها على العدد الذي هو تسعة وثلثون ليتم لنا بناء
السطح الاعظم بما نقص من زوايا ه الاربع لان
كل عدد يضرب ربعه في مثله ثم في اربعة يكون
مثل ضرب نصفه في مثله فاستغنينا بضرب
نصف الاجذار في مثلها عن الربع في مثله ثم في اربعة
وهذا صورته

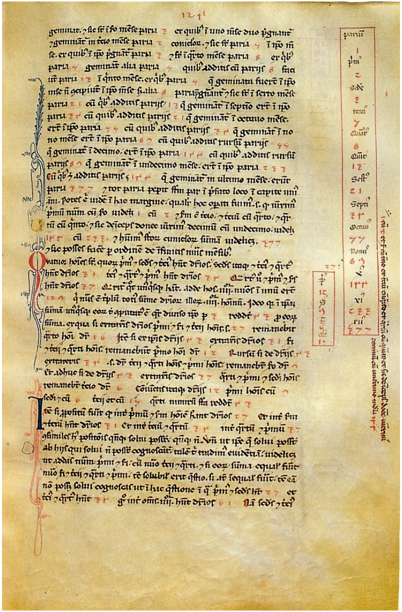
وله ايضا صورة اخرى تؤدي الى هذا وهي سطح
اب وهو المال فاردنا ان نزيد عليه مثل عشرة

An algorithm for case (4) on the previous slide

# Leonardo of Pisa (Fibonacci) (c. 1175–c. 1240/50)

*Liber abaci* (or *Liber abbaci*), Pisa, 1202:

- included al-Khwārizmi's recipes
- geometrical demonstrations and lots of examples
- didn't go very far beyond predecessors, but began transmission of Islamic ideas to Europe



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- passed in rhyme to Girolamo Cardano (1539)

# Cubic equations (2)

$$x^3 + px = q$$

*When the cube with the things next after*
*Together equal some number apart*
*Find two others that by this differ*
*And this you will keep as a rule*
*That their product will always be equal*
*To a third cubed of the number of things*
*The difference then in general between*
*The sides of the cubes subtracted well*
*Will be your principal thing.*

(Tartaglia, 1546; see: *Mathematics emerging*, §12.1.1)

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**NB:** In an equation $y^3 + ay^2 + by + c = 0$ we can put $y = x - \frac{a}{3}$ to remove the square term, so this solution is general.

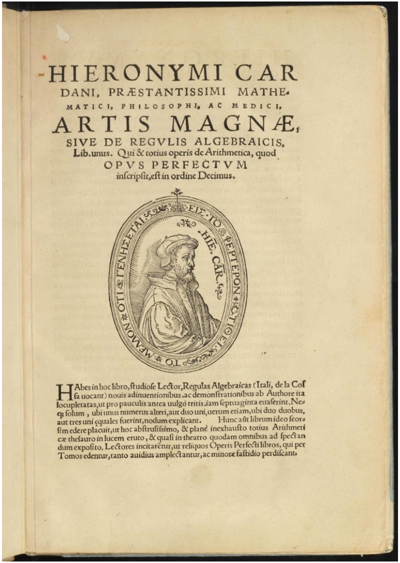
# Cubic equations (4)

In modern terms, one of the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

with similar expressions (in radicals) for the remaining two roots

# Cardano's *Ars magna, sive de regulis algebraicis* (1545)



HIERONYMI CAR
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MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarētur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

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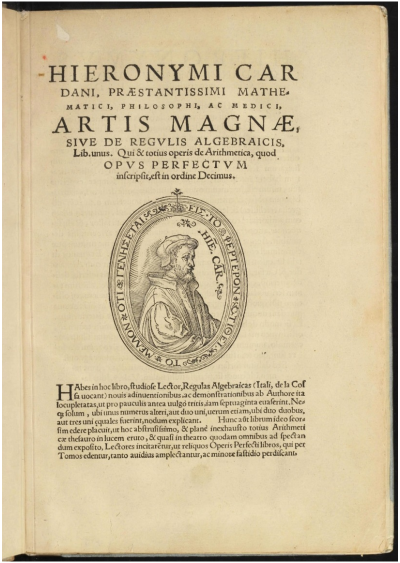

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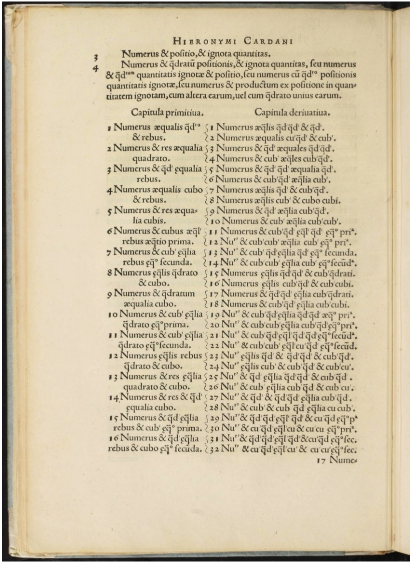
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inſcripſit, eſt in ordine Decimus.



HAbes in hoc libro, ſtudioſe Lector, Regulas Algebraicas (Itali, de la Coſſa uocant) nouis adinuentionibus, ac demonſtrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam ſeptuaginta euaſerint. Neq; ſolum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni ęquales fuerint, nodum explicant. Hunc aũt librum ideo ſeorſim edere placuit, ut hoc abſtruſiſſimo, & planè inexhauſto totius Arithmeticæ theſauro in lucem eruto, & quaſi in theatro quodam omnibus ad ſpectandum expoſito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore faſtidio perdiſcant.





3 Numerus & poſitio, & ignota quantitas.
4 Numerus & q̄dratũ poſitionis, & ignota quantitas, ſeu numerus & q̄dtum quantitatis ignotæ & poſitio, ſeu numerus cũ q̄dto poſitionis quantitatis ignotæ, ſeu numerus & productum ex poſitione in quantitatem ignotam, cum altera earum, uel cum q̄drato unius earum.

| Capitula primitiua. | Capitula deriuatiua. |
| --- | --- |
| 1 Numerus æqualis q̄dto & rebus. | 1 Numerus æq̄lis q̄d'q̄d' & q̄d'. <br> 2 Numerus æqualis cu'q̄d' & cub'. |
| 2 Numerus & res æqualia quadrato. | 3 Numerus & q̄d' æquales q̄d'q̄d'. <br> 4 Numerus & cub' æq̄les cub'q̄d'. |
| 3 Numerus & q̄d' ęqualia rebus. | 5 Numerus & q̄d' q̄d' æqualia q̄d'. <br> 6 Numerus & cub'q̄d' æq̄lia cub'. |
| 4 Numerus æqualis cubo & rebus. | 7 Numerus æq̄lis q̄d' & cub'q̄d'. <br> 8 Numerus æq̄lis cub' & cubo cubi. |
| 5 Numerus & res æqualia cubis. | 9 Numerus & q̄d' æq̄lia cub'q̄d'. <br> 10 Numerus & cub' æq̄lia cub'cub'. |
| 6 Numerus & cubus æq̄l' rebus æq̄tio prima. | 11 Numerus & cub'q̄d' ęq̄l' q̄d' ęq̄o pria. <br> 12 Nur' & cub'cub' æq̄lia cub' ęq̄o pria. |
| 7 Numerus & cub' ęq̄lia rebus ęq̄o ſecunda. | 13 Nur' & cub' q̄d'ęq̄lia q̄d' ęq̄o ſecunda. <br> 14 Nur' & cub'cub' ęq̄lia cub' ęq̄o ſecũda. |
| 8 Numerus ęq̄lis q̄drato & cubo. | 15 Numerus ęq̄lis q̄d'q̄d' & cub'q̄drati. <br> 16 Numerus ęq̄lis cub'q̄d' & cub'cubi. |
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| 10 Numerus & cub' ęq̄lia q̄drato ęq̄o prima. | 19 Nur' & cub'q̄d'ęq̄lia q̄d'q̄d' æq̄o pria. <br> 20 Nur' & cub'cub'ęq̄lia cub'q̄d'ęq̄o pria. |
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| 12 Numerus ęq̄lis rebus q̄drato & cubo. | 23 Nur' ęq̄lis q̄d' & q̄d'q̄d' & cub'q̄d'. <br> 24 Nur' ęq̄lis cub' & cub'q̄d' & cub'cu'. |
| 13 Numerus & res ęq̄lia quadrato & cubo. | 25 Nur' & q̄d' ęq̄lia q̄d'q̄d' & cub'q̄d'. <br> 26 Nur' & cub' ęq̄lia cub'q̄d & cub'cu'. |
| 14 Numerus & res & q̄d' ęqualia cubo. | 27 Nur' & q̄d' & q̄d'q̄d' ęq̄lia cub'q̄d'. <br> 28 Nur' & cub' & cub' q̄d' ęq̄lia cu cub'. |
| 15 Numerus & q̄d' ęq̄lia rebus & cub' ęq̄o prima. | 29 Nur' & q̄d'q̄d' ęq̄l' q̄d' & cu'q̄d ęq̄o pa <br> 30 Nur' & cu'q̄d' ęq̄l'cu & cu'cu ęq̄o pria. |
| 16 Numerus & q̄d' ęq̄lia rebus & cubo ęq̄o ſecũda. | 31 Nur' & q̄d'q̄d' ęq̄l' q̄d' & cu'q̄d ęq̄o ſec. <br> 32 Nur' & cu'q̄d' ęq̄l'cu' & cu'cu' ęq̄o ſec. |

17 Nume-

# Cardano on the cubic



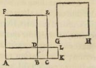
relinquitur prima 6 m: R 30$\frac{6}{7}$, hæ autem quantitates proportionales ſunt, & quadratum ſecundæ eſt æquale duplo producti ſecundæ in primam, cum quadruplo primæ, ut proponebatur.

## De cubo & rebus æqualibus numero. Cap. XI.

Scipio Ferreus Bononienſis iam annis ab hinc triginta fermè capitulum hoc inuenit, tradidit uero Anthonio Mariæ Florido Veneto, qui cũ in certamen cũ Nicolao Tartalea Brixellenſe aliquando ueniſſet, occaſionem dedit, ut Nicolaus inuenerit & ipſe, qui cum nobis rogantibus tradidiſſet, ſuppreſſa demonſtratione, freti hoc auxilio, demonſtrationem quæſiuimus, eamq; in modos, quod difficillimum fuit, redactam ſic ſubiecimus.

DEMONSTRATIO.

Sit igitur exempli cauſa cubus G H & ſexcuplum lateris G H æquale 20, & ponam duos cubos A E & C L, quorum differentia ſit 20, ita quod productum A C lateris, in C K latus, ſit 2, tertia ſcilicet numeri rerum pars, & abſcindam C B, æqualem C K, dico, quod ſi ita fuerit, lineam A B reſiduum, eſſe æqualem G H, & ideo rei æſtimationem, nam de G H iam ſupponebatur, quod ita eſſet, perficiam igitur per modum primi ſuppoſiti 6ⁱ capituli huius libri, corpora D A, D C, D E D F, ut per D C intelligamus cubum B C, per




D F cubum A B, per D A triplum C B in quadratum A B, per D E triplum A B in quadratũ B C. quia igitur ex A C in C K fit 2, ex A C in C K ter, fiet 6 numerus rerum, igitur ex A B in triplum A C in C K fiunt 6 res A B, ſeu ſexcuplum A B, quare triplum producti ex A B, B C, A C, eſt ſexcuplum A B, at uero differentia cubi A C, à cubo C K, & exiſtenti à cubo B C ei æq̃le ex ſuppoſito, eſt 20, & ex ſuppoſito primo 6ⁱ capituli, eſt aggregatum corporum D A, D E, D F, tria igitur hæc corpora ſunt 20, poſita uero B C m: cubus A B, æqualis eſt cubo A C, & triplo A C in quadratum C B, & cubo B C m: & triplo B C in quadratum A C m: per demonſtrata illic, differentia autem tripli B C in quadratum A C, à triplo A C in quadratum B C eſt productum A B, B C, A C, quare cum hoc, ut demonſtratum eſt, æquale ſit ſexcuplo A B, igitur addito ſexcuplo A B, ad id quod fit ex A C in quadratum B C ter, fiet triplum B C in quadratum A C, cum igitur B C ſit m: iam oſtenſum eſt, quod productum C B

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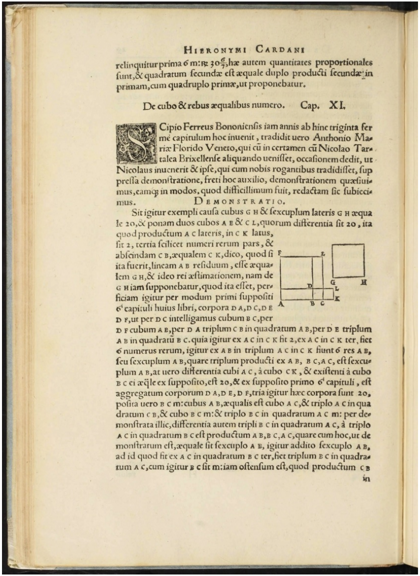

HIERONYMI CARDANI

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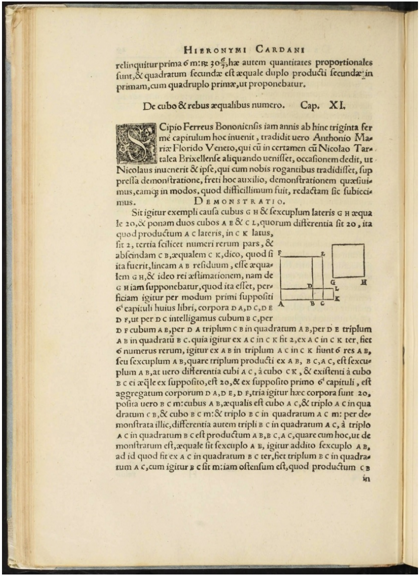

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- General solution (to particular case), rather than example to be followed

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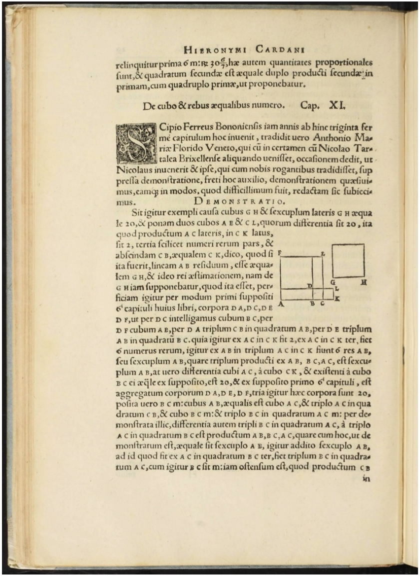

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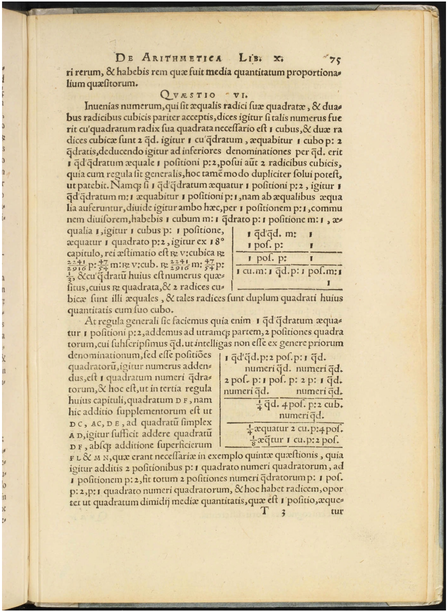
- Make substitution $x = y - \frac{a}{3}$ in $y^3 + ax^2 + bx + c = d$ to suppress square term and obtain equation of the form $x^3 + px = q$ — manipulation of equations prior to solution

# Quartic equations (1)

General solution discovered (again on a case-by-case basis) by Lodovico Ferrari (c. 1540) and published by Cardano, in the form of worked examples, alongside solution of cubic



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ri rerum, & habebis rem quæ fuit media quantitatum proportionalium quæsitorum.

QVÆSTIO VI.

Inuenias numerum, qui sit æqualis radici suæ quadratæ, & duabus radicibus cubicis pariter acceptis, dices igitur si talis numerus fuerit cu'quadratum radix sua quadrata necessario est 1 cubus, & duæ radices cubicæ sunt 2 q̄d. igitur 1 cu'q̄dratum, æquabitur 1 cubo p: 2 q̄dratis, deducendo igitur ad inferiores denominationes per q̄d. erit 1 q̄d'q̄dratum æquale 1 positioni p:2, posui aūt 2 radicibus cubicis, quia cum regula sit generalis, hoc tamē modo dupliciter solui potest, ut patebit. Namq; si 1 q̄d'q̄dratum æquatur 1 positioni p:2, igitur 1 q̄d'q̄dratum m:1 æquabitur 1 positioni p:1, nam ab æqualibus æqualia auferuntur, diuide igitur ambo hæc, per 1 positionem p:1, communem diuisorem, habebis 1 cubum m:1 q̄drato p:1 positione m:1, æqualia 1, igitur 1 cubus p: 1 positione, æquatur 1 quadrato p:2, igitur ex 18° capitulo, rei æstimatio est R v:cubica R $\frac{2241}{2916}$ p: $\frac{47}{54}$ m:R v:cub. R $\frac{2241}{2916}$ m: $\frac{47}{54}$ p: $\frac{1}{3}$, & cu'q̄dratū huius est numerus quæsitus, cuius R quadrata, & 2 radices cubicæ sunt illi æquales, & tales radices sunt duplum quadrati huius quantitatis cum suo cubo.

1 q̄d'q̄d. m: 1
1 pos. p: 1
1 pos. p: 1
1 cu.m: 1 q̄d.p: 1 pos.m: 1
1

At regula generali sic faciemus quia enim 1 q̄d'q̄dratum æquatur 1 positioni p:2, addemus ad utramq; partem, 2 positiones quadratorum, cui subscripsimus q̄d. ut intelligas non esse ex genere priorum denominationum, sed esse positiões quadratorū, igitur numerus addendus, est 1 quadratum numeri q̄dratorum, & hoc est, ut in tertia regula huius capituli, quadratum D F, nam hic additio supplementorum est ut D C, AC, D E, ad quadratū simplex A D, igitur sufficit addere quadratū D F, absq; additione superficierum F L & M N, quæ erant necessariæ in exemplo quintæ quæstionis, quia igitur additis 2 positionibus p: 1 quadrato numeri quadratorum, ad 1 positionem p:2, fit totum 2 positiones numeri q̄dratorum p: 1 pos. p:2, p:1 quadrato numeri quadratorum, & hoc habet radicem, oportet ut quadratum dimidij mediæ quantitatis, quæ est 1 positio, æquetur

1 q̄d'q̄d.p:2 pos.p: 1 q̄d.
numeri q̄d. numeri q̄d.
2 pos. p: 1 pos. p: 2 p: 1 q̄d.
numeri q̄d. numeri q̄d.
$\frac{1}{4}$ q̄d. 4 pos. p:2 cub.
numeri q̄d.
$\frac{1}{4}$ æquatur 2 cu.p:4 pos.
$\frac{1}{8}$ æq̄tur 1 cu.p:2 pos.

T 3 tur

# Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

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$$8y^3 + 4py^2 + 8ry + (4pr - q^2) = 0.$$

So the problem is reduced to solving a cubic equation and then a quadratic.

**NB:** In an equation $y^4 + ay^3 + by^2 + cy + d = 0$ we can put $y = x - \frac{a}{4}$ to remove the cube term, so this solution is general.

# Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

http://planetmath.org/QuarticFormula

Cardano's *Ars Magna* may also be found online here

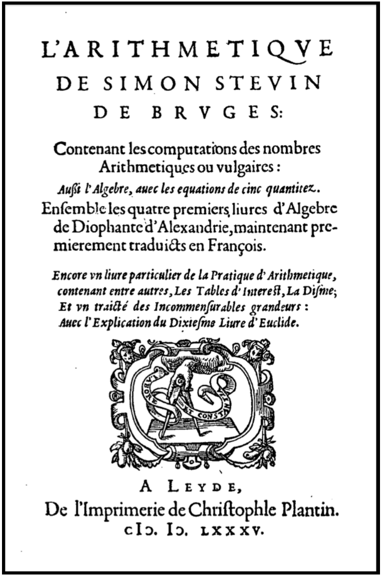
# Further 16th-century developments


Rafael Bombelli, *L'algebra* (1572):

- heavily influenced by Cardano
- equation solving, new notation
- exploration of complex numbers

  [to be dealt with in a later lecture]

# Further 16th-century developments



L'ARITHMETIQVE
DE SIMON STEVIN
DE BRVGES:

Contenant les computations des nombres Arithmetiques ou vulgaires:

*Aussi l'Algebre, auec les equations de cinc quantitez.*

Ensemble les quatre premiers liures d'Algebre de Diophante d'Alexandrie, maintenant premierement traduicts en François.

*Encore vn liure particulier de la Pratique d'Arithmetique, contenant entre autres, Les Tables d'Interest, La Disme; Et vn traicté des Incommensurables grandeurs: Auec l'Explication du Dixiesme Liure d'Euclide.*

A LEYDE,
De l'Imprimerie de Christophle Plantin.
cIↃ. IↃ. LXXXV.

Simon Stevin, *L'arithmetique ... aussi l'algebre* (1585):

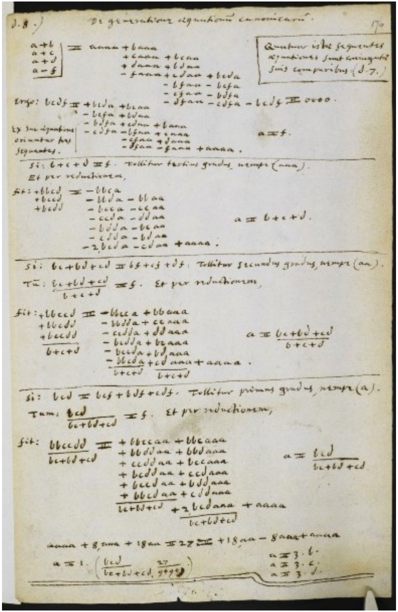
- heavily influenced by Cardano through Bombelli
- appended his treatise on decimal notation

# Further 16th-century developments

François Viète (1590s):

- links between algebra and geometry
- (algebra as 'analysis' or 'analytic art')
- notation [recall Lecture III]
- numerical methods for solving equations

# Thomas Harriot (c. 1600)

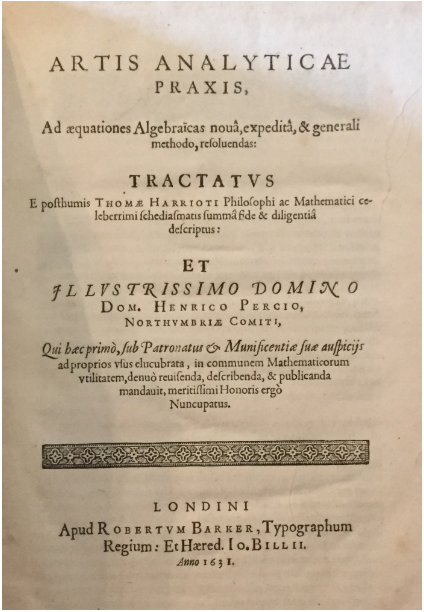


Add MS 6783 f. 176

Note:

- notation [see lecture III];
- appearance of polynomials as products of linear factors.

# Thomas Harriot (1631)



ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraïcas nouâ, expeditâ, & generali methodo, resoluendas:

TRACTATVS

E posthumis THOMÆ HARRIOTI Philosophi ac Mathematici celeberrimi schediasmatis summâ fide & diligentiâ descriptus:

ET

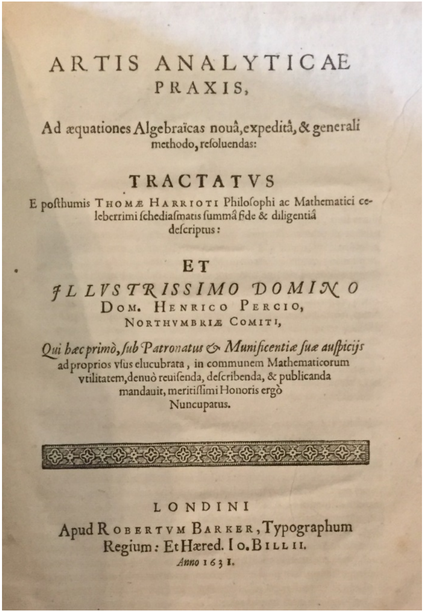
ILLVSTRISSIMO DOMINO
DOM. HENRICO PERCIO,
NORTHVMBRIÆ COMITI,

*Qui hæc primò, sub Patronatus & Munificentiæ suæ auspicijs* ad proprios vsus elucubrata, in communem Mathematicorum vtilitatem, denuò reuisenda, describenda, & publicanda mandauit, meritissimi Honoris ergò Nuncupatus.

LONDINI
Apud ROBERTVM BARKER, Typographum Regium: Et Hæred. Io. BILLII.
*Anno* 1631.

Some of Harriot's ideas found their way into his *Artis analyticae praxis* (*The practice of the analytic art*), published posthumously in 1631

# Thomas Harriot (1631)



ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraïcas nouâ, expeditâ, & generali methodo, reſoluendas:

TRACTATVS

E poſthumis THOMÆ HARRIOTI Philoſophi ac Mathematici celeberrimi ſchediaſmatis ſummâ fide & diligentiâ deſcriptus:

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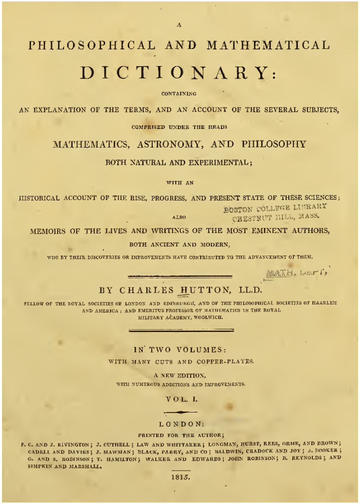
But editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

# Commentary on Harriot



A

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BY CHARLES HUTTON, LL.D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH, AND OF THE PHILOSOPHICAL SOCIETIES OF HAARLEM AND AMERICA; AND EMERITUS PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

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1815.

Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

*He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; ...*