# BO1 History of Mathematics
# Lecture IX
# Classical algebra: equation solving 1800BC – AD1800
# Part 2: The theory of equations

MT 2020 Week 5

# Algebra in the 17th century

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From 1600 onwards, ‘algebra’ as a set of recipes and techniques began to diverge in two (linked) directions:

- ‘algebra’ as a tool or a language (a.k.a. ‘analysis’ or the ‘analytic art’)
- ‘algebra’ as an object of study in its own right (the ‘theory of equations’)

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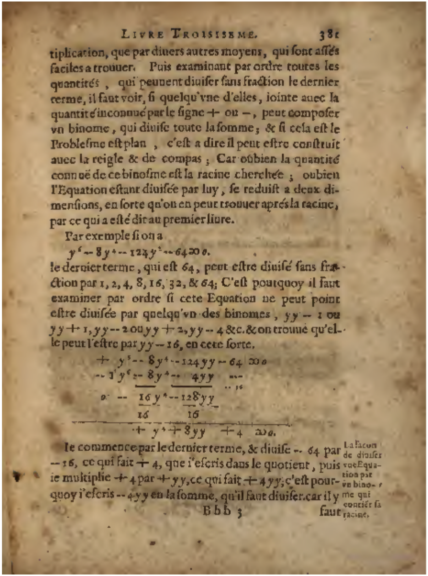
  has at most 3 positive roots and at most one negative;
- can always make a transformation to remove the second-highest term.

# Descartes on cubics




tiplication, que par diuers autres moyens, qui sont assés faciles a trouuer. Puis examinant par ordre toutes les quantités, qui peuuent diuiser sans fraction le dernier terme, il faut voir, si quelqu'vne d'elles, iointe auec la quantité inconnuë par le signe + ou –, peut composer vn binome, qui diuise toute la somme; & si cela est le Problesme est plan, c'est a dire il peut estre construit auec la reigle & de compas; Car oûbien la quantité connuë de ce binosme est la racine cherchée; oubien l'Equation estant diuisée par luy, se reduist a deux dimensions, en sorte qu'on en peut trouuer aprés la racine, par ce qui a esté dit au premier liure.

Par exemple si on a

$y^6 - 8y^4 - 124y^2 - 64 \propto 0.$

le dernier terme, qui est 64, peut estre diuisé sans fraction par 1, 2, 4, 8, 16, 32, & 64; C'est pourquoy il faut examiner par ordre si cete Equation ne peut point estre diuisée par quelqu'vn des binomes, $yy - 1$ ou $yy + 1$, $yy - 2$ ou $yy + 2$, $yy - 4$ &c. & on trouue qu'elle peut l'estre par $yy - 16$, en cete sorte.

$+ y^6 - 8y^4 - 124yy - 64 \propto 0$

$- 1y^6 - 8y^4 - 4yy \quad - 16$

$0 - \frac{16y^4}{16} - \frac{128yy}{16}$

$+ y^4 + 8yy + 4 \propto 0.$

Ie commence par le dernier terme, & diuise $-64$ par $-16$, ce qui fait $+4$, que i'escris dans le quotient, puis ie multiplie $+4$ par $+yy$, ce qui fait $+4yy$; c'est pourquoy i'escris $-4yy$ en la somme, qu'il faut diuiser. car il y faut

La façon de diuiser vne Equation par vn binome qui contient sa racine.



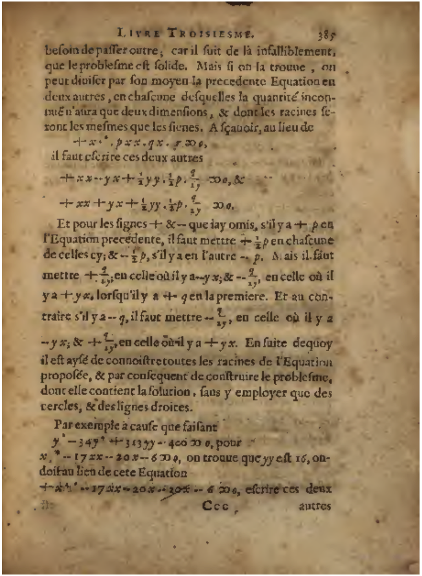
Search for roots of a cubic by examining the factors of the constant term:

if $\alpha$ is such a factor, test whether $x - \alpha$ divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

# Descartes on quartics



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besoin de passer outre; car il suit de là infalliblement, que le problesme est solide. Mais si on la trouue, on peut diuiser par son moyen la precedente Equation en deux autres, en chascune desquelles la quantité inconnuë n'aura que deux dimensions, & dont les racines seront les mesmes que les siennes. A sçauoir, au lieu de

$+x^4 * .pxx.qx.r \propto 0,$

il faut escrire ces deux autres

$+xx - yx + \frac{1}{2}yy . \frac{1}{2}p . \frac{q}{2y} \propto 0$, &

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Par exemple à cause que faisant

$y^6 - 34y^4 + 313yy - 400 \propto 0$, pour

$x^4 * -17xx - 20x - 6 \propto 0$, on trouue que $yy$ est 16, on doit au lieu de cete Equation

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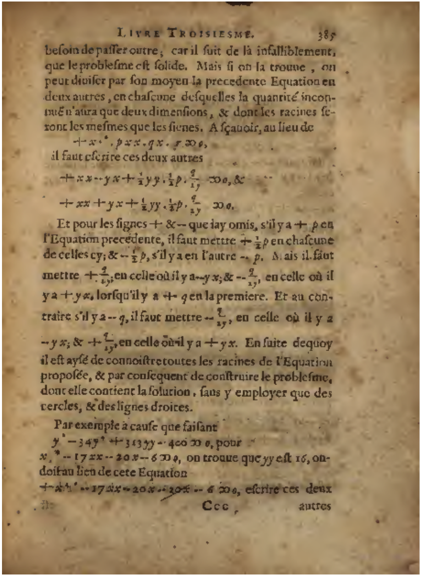



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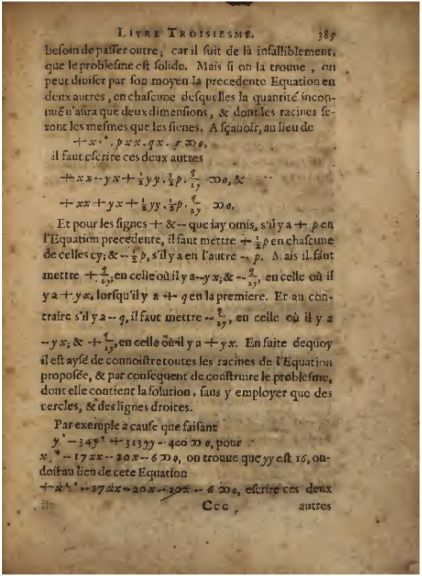



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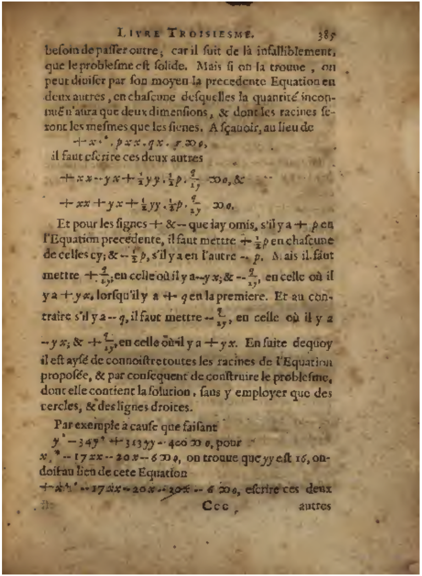



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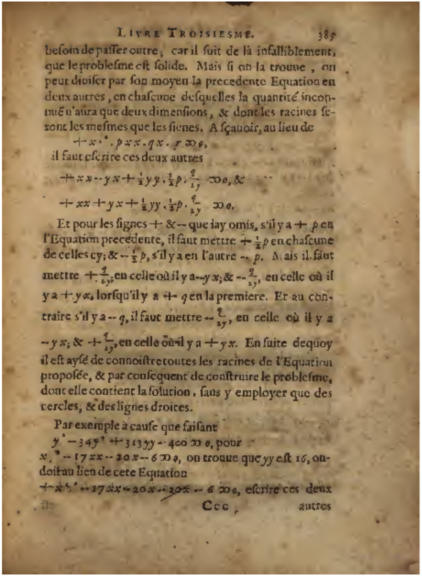



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As in Ferrari's/Cardano's method: a quartic is reduced to a cubic

# Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions in radicals, i.e., solutions constructed from the coefficients of a given polynomial equation via $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, $\ldots$

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So did anything interesting happen in algebra during the 17th and 18th centuries?

# A typical 20th-century view

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

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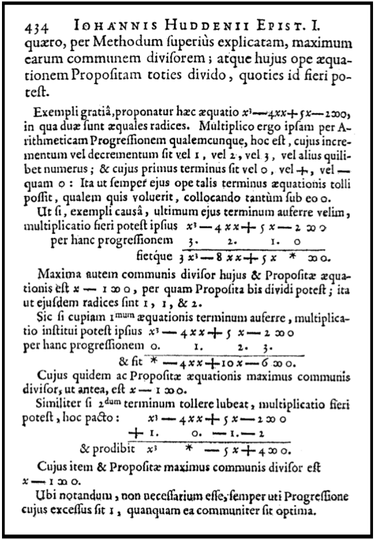
Fair point? Or not?

# Some 17th-century developments: Hudde's rule (1657)

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' *La géométrie*:



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quæro, per Methodum superiùs explicatam, maximum earum communem divisorem; atque hujus ope æquationem Propositam toties divido, quoties id fieri potest.

Exempli gratiâ, proponatur hæc æquatio $x^3 - 4xx + 5x - 2 \infty 0$, in qua duæ sunt æquales radices. Multiplico ergo ipsam per Arithmeticam Progressionem qualemcunque, hoc est, cujus incrementum vel decrementum sit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus sit vel 0, vel +, vel — quam 0: Ita ut semper ejus ope talis terminus æquationis tolli possit, qualem quis voluerit, collocando tantùm sub eo 0.

Ut si, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri potest ipsius

$$\begin{array}{llllll} & x^3 & -\,4xx & +\,5x & -\,2 & \infty\, 0 \\ \text{per hanc progressionem} & 3. & 2. & 1. & 0 & \\ \hline \text{fietque} & 3x^3 & -\,8xx & +\,5x & * & \infty\, 0. \end{array}$$

Maxima autem communis divisor hujus & Propositæ æquationis est $x - 1 \infty 0$, per quam Proposita bis dividi potest; ita ut ejusdem radices sint 1, 1, & 2.

Sic si cupiam $1^{\text{mum}}$ æquationis terminum auferre, multiplicatio institui potest ipsius

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Cujus quidem ac Propositæ æquationis maximus communis divisor, ut antea, est $x - 1 \infty 0$.

Similiter si $2^{\text{dum}}$ terminum tollere lubeat, multiplicatio fieri potest, hoc pacto:

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Ubi notandum, non necessarium esse, semper uti Progressione cujus excessus sit 1, quanquam ea communiter sit optima.

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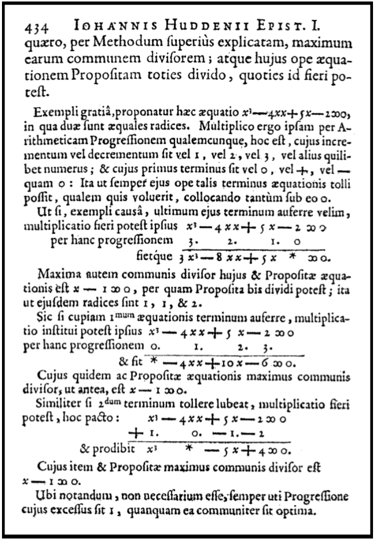



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fietque $3x^3 - 8xx + 5x \quad * \quad \propto 0.$

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& fit $* \; - 4xx + 10x - 6 \propto 0.$

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$x^3 - 4xx + 5x - 2 = 0$ has a double root $x = 1$;

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Ut ſi, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri poteſt ipſius $x^3 - 4xx + 5x - 2 \propto 0$
per hanc progreſſionem 3. 2. 1. 0
fietque $3x^3 - 8xx + 5x \; * \propto 0$.

Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \propto 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

Sic ſi cupiam 1mum æquationis terminum auferre, multiplicatio inſtitui poteſt ipſius $x^3 - 4xx + 5x - 2 \propto 0$
per hanc progreſſionem 0. 1. 2. 3.
& fit $* - 4xx + 10x - 6 \propto 0$.

Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \propto 0$.

Similiter ſi 2dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \propto 0$
+1. 0. —1. —2
& prodibit $x^3 \; * - 5x + 4 \propto 0$.

Cujus item & Propoſitæ maximus communis diviſor eſt $x - 1 \propto 0$.

Ubi notandum, non neceſſarium eſſe, ſemper uti Progreſſione cujus exceſſus ſit 1, quanquam ea communiter ſit optima.

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' *La géométrie*:

$x^3 - 4xx + 5x - 2 = 0$ has a double root $x = 1$;

multiply the terms of the equation by numbers in arithmetic progression:

# Some 17th-century developments: Hudde's rule (1657)



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Exempli gratiâ, proponatur hæc æquatio $x^3 - 4xx + 5x - 2 \infty 0$, in qua duæ ſunt æquales radices. Multiplico ergo ipſam per Arithmeticam Progreſſionem qualemcunque, hoc eſt, cujus incrementum vel decrementum ſit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus ſit vel 0, vel +, vel — quam 0: Ita ut ſemper ejus ope talis terminus æquationis tolli poſſit, qualem quis voluerit, collocando tantùm ſub eo 0.

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$$\begin{array}{rcccc} & x^3 - & 4xx + & 5x - & 2 \infty 0 \\ \text{per hanc progreſſionem} & 3. & 2. & 1. & 0 \\ \hline \text{fietque} & 3x^3 - & 8xx + & 5x & * \infty 0. \end{array}$$

Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \infty 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

Sic ſi cupiam 1mum æquationis terminum auferre, multiplicatio inſtitui poteſt ipſius

$$\begin{array}{rcccc} & x^3 - & 4xx + & 5x - & 2 \infty 0 \\ \text{per hanc progreſſionem} & 0. & 1. & 2. & 3. \\ \hline \text{\& fit} & * & -4xx + & 10x - & 6 \infty 0. \end{array}$$

Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \infty 0$.

Similiter ſi 2dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto:

$$\begin{array}{rcccc} & x^3 - & 4xx + & 5x - & 2 \infty 0 \\ & +1. & 0. & -1. & -2 \\ \hline \text{\& prodibit} & x^3 & * & -5x + & 4 \infty 0. \end{array}$$

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Ut ſi, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$
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Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \infty 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

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per hanc progreſſionem $0. \quad 1. \quad 2. \quad 3.$
& fit $* \quad -4xx + 10x - 6 \infty 0.$

Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \infty 0$.

Similiter ſi 2dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \infty 0$
$+1. \quad 0. \quad -1. \quad -2$
& prodibit $x^3 \quad * \quad -5x + 4 \infty 0.$

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Ut ſi, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$

per hanc progreſſionem $3. \quad 2. \quad 1. \quad 0$

fietque $3x^3 - 8xx + 5x \quad * \quad \infty 0$.

Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \infty 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

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Similiter ſi 2dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \infty 0$

$+1. \quad 0. \quad -1. \quad -2$

& prodibit $x^3 \quad * \quad -5x + 4 \infty 0$.

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as does $-4xx + 10x - 6 = 0$.

(Modern form of rule: if $r$ is a double root of $f(x) = 0$, then it is a root of $f'(x) = 0$ also.)

See *Mathematics emerging*, §12.2.2.

# Some 17th-century developments: Tschirnhaus transformations (1683)



*METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione,*
*per D. T.*

EX Geometria Dn. Des Cartes' notum est, qua ratione semper secundus terminus ex data æquatione possit auferri; quoad plures terminos intermedios auferendos, hactenus nihil inventum vidi in Arte Analytica, imo non paucos offendi, qui crediderunt, id nulla arte perfici posse. Quapropter hic quædam circa hoc negotium aperire constitui, verum saltem pro iis, qui Artis Analyticæ apprime gnari, cum aliis tam brevi explicatione vix satisfieri possit: reliqua, quæ hic desiderari possent, alii tempori reservans.

Primo itaque loco, ad hoc attendendum; sit data aliqua æquatio cubica $x^3 - pxx + qx - r = 0$, in qua $x$ radices hujus æquationis designat; $p, q, r$, cognitas quantitates repræsentant: ad auferendum jam secundum terminum supponatur $x = y + a$; jam ope harum duarum æquationum inveniatur tertia, ubi quantitas $x$ absit, & erit

$$\begin{array}{llll} y^3 & +3ayy & +3aay & +a^3 = 0 \\ & -pyy & -2pay & -paa \\ & & +qy & +qa \\ & & & -r \end{array}$$

Ponatur nunc secundus terminus æqualis nihilo (quia hunc auferre nostra intentio) eritque $3ayy - pyy = 0$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum secundum terminum in æquatione Cubica, supponendum esse loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum sunt, nec hic referuntur aliam ob causam, quam quia sequentia admodum illustrant, dum hisce bene intellectis, eo facilius, quæ modo proponam, capientur.

Sint jam secundo in æquatione data auferendi duo termini: dico, quod supponendum sit, $xx = bx + y + a$; si tres, $x^3 = cxx + bx + y + a$; si quatuor, $x^4 = dx^3 + cxx + bx + y + a$, atque sic in infinitum. Vocabo autem has *æquationes assumtas*, ut eas distinguam ab æquatione, quæ ut data consideratur. Ratio autem horum est: quod eadem ratione, prout ope æquationis $x = y + a$ saltem unicus terminus poterat auferri, quia nimirum unica saltem indeterminata hic existit $a$, sic eadem ratione ope hujus $xx = bx + y + a$, non nisi duo termini possunt auferri, quia duæ indeterminatæ $a$ & $b$ adsunt;

For an equation $x^3 - px^2 + qx - r = 0$

# Some 17th-century developments: Tschirnhaus transformations (1683)



*METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione, per D. T.*

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Primo itaque loco, ad hoc attendendum; ſit data aliqua æquatio cubica $x^3 - pxx + qx - r = o$, in qua $x$ radices hujus æquationis deſignat; $p, q, r$, cognitas quantitates repræſentant: ad auferendum jam ſecundum terminum ſupponatur $x = y + a$; jam ope harum duarum æquationum inveniatur tertia, ubi quantitas $x$ abſit, & erit

$$\begin{array}{llll} y^3 + 3ayy & + 3aay & + a^3 & = o \\ \quad - pyy & - 2pay & - paa & \\ & + qy & + qa & \\ & & - r & \end{array}$$

Ponatur nunc ſecundus terminus æqualis nihilo (quia hunc auferre noſtra intentio) eritque $3ayy - pyy = o$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum ſecundum terminum in æquatione Cubica, ſupponendum eſſe loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum ſunt, nec hic referuntur aliam ob cauſam, quam quia ſequentia admodum illuſtrant, dum hiſce bene intellectis, eo facilius, quæ modo proponam, capientur.

Sint jam ſecundo in æquatione data auferendi duo termini: dico, quod ſupponendum ſit, $xx = bx + y + a$; ſi tres, $x^3 = cxx + bx + y + a$; ſi quatuor, $x^4 = dx^3 + cxx + bx + y + a$, atque ſic in infinitum. Vocabo autem has *æquationes aſſumtas*, ut eas diſtinguam ab æquatione, quæ ut data conſideratur. Ratio autem horum eſt: quod eadem ratione, prout ope æquationis $x = y + a$ ſaltem unicus terminus poterat auferri, quia nimirum unica ſaltem indeterminata hic exiſtit $a$, ſic eadem ratione ope hujus $xx = bx + y + a$, non niſi duo termini poſſunt auferri, quia duæ indeterminatæ $a$ & $b$ adſunt;

For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)

# Some 17th-century developments: Tschirnhaus transformations (1683)

204 ACTA ERUDITORUM

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*per D. T.*

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$$\begin{array}{llll} y^3 & +3ayy & +3aay & +a^3 = 0 \\ & -pyy & -2pay & -paa \\ & & +qy & +qa \\ & & & -r \end{array}$$

Ponatur nunc ſecundus terminus æqualis nihilo (quia hunc auferre noſtra intentio) eritque $3ayy - pyy = 0$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum ſecundum terminum in æquatione Cubica, ſupponendum eſſe loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum ſunt, nec hic referuntur aliam ob cauſam, quam quia ſequentia admodum illuſtrant, dum hiſce bene intellectis, eo facilius, quæ modo proponam, capientur.

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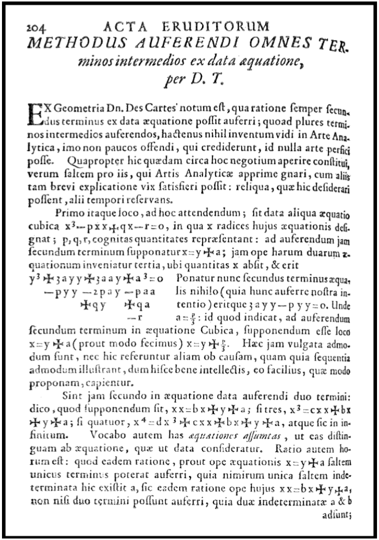
For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)
- can we remove both the middle terms?

# Some 17th-century developments: Tschirnhaus transformations (1683)



204 ACTA ERUDITORUM

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$$\begin{array}{llll} y^3 + 3ayy + 3aay + a^3 = 0 \\ \quad - pyy - 2pay - paa \\ \qquad\quad + qy \quad + qa \\ \qquad\qquad\quad\; - r \end{array}$$

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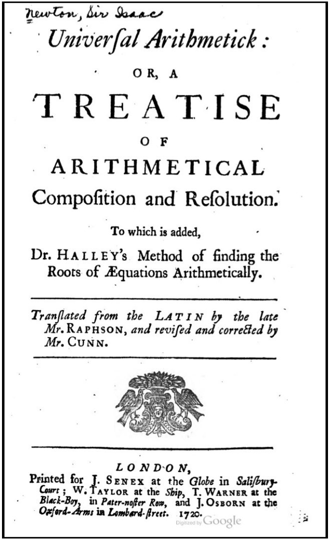
For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)
- can we remove both the middle terms?
- to remove two terms put $x^2 = bx + y + a$

See *Mathematics emerging*, §12.2.3.

# An 18th-century development: Newton's *Arithmetica universalis* (1707)


Newton, Sir Isaac

*Univerſal Arithmetick:*

OR, A

TREATISE

OF

ARITHMETICAL

Compoſition and Reſolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

*Tranſlated from the* LATIN *by the late Mr.* RAPHSON, *and reviſed and corrected by Mr.* CUNN.

LONDON,

Printed for J. SENEX at the *Globe* in *Saliſbury-Court*; W. TAYLOR at the *Ship*, T. WARNER at the *Black-Boy*, in *Pater-noſter Row*, and J. OSBORN at the *Oxford-Arms* in *Lombard-ſtreet*. 1720.



Rules for sums of powers of roots of

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \cdots = 0$$

$$\begin{aligned}
\text{sum of roots} &= p \\
\text{sum of roots}^2 &= pa - 2q \\
\text{sum of roots}^3 &= pb - qa + 3r \\
\text{sum of roots}^4 &= pc - qb + ra - 4s
\end{aligned}$$

# Developments of the 17th and 18th centuries

- Symbolic notation

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- Understanding of the structure of polynomials

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# Developments of the 17th and 18th centuries

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- The leaving behind of geometric intuition?

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- possibly further reducible to $(n - 2)!$

# Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- quadratics: the well-known solution
- cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

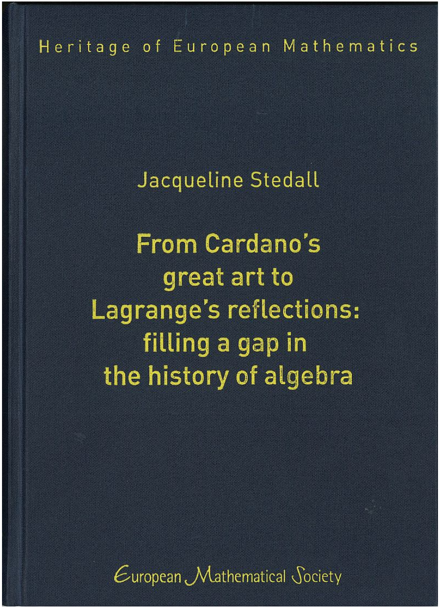
seeking to identify a uniform method that could be extended to higher degree

# A typical 20th-century view revisited

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

# Filling a gap in the history of algebra (2011)




*The hitherto untold story of the slow and halting journey from Cardano's solution recipes to Lagrange's sophisticated considerations of permutations and functions of the roots of equations* ... [Preface]

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