BO1 History of Mathematics
Lecture X
The 19th-century beginnings of ‘modern algebra’
Part 1: Resolvents and permutations

MT 2020 Week 5

Summary

Part 1

- Lagrange's ideas (1770/71)
- Cauchy and substitutions (1815)
- 'Classical age' of theory of equations 'ends' (1799–1826)

Part 2

- The invention of groups by Galois and Cauchy

Part 3

- 'Symbolical algebra'
- Groups, rings, and fields: the emergence of 'modern algebra' (1854–1900)

‘Modern’ or ‘abstract’ algebra

19th century: emergence of mathematics whose subject-matter is no longer space or number:

- permutations
- abstract structures (groups, rings, fields, ...)
- linear algebra [see Lecture XIV]

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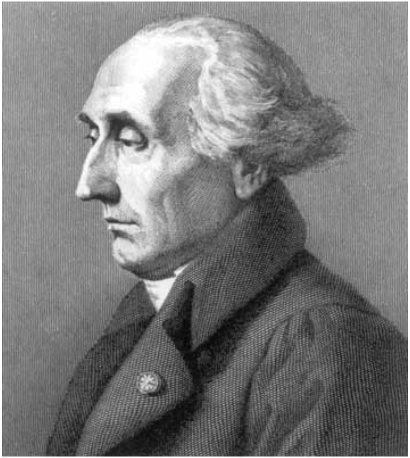
J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/71):



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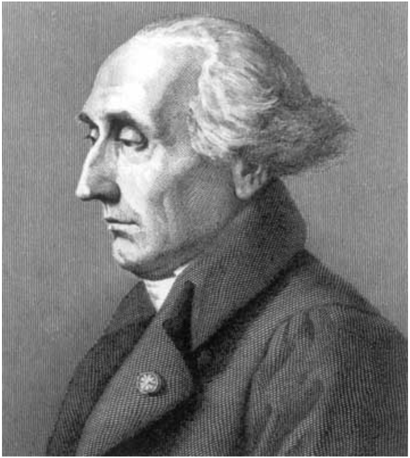
Asserted that there had been little advance in equation-solving since Cardano, but that there was little left to do

Examined all known methods of solving cubics and quartics

Found that in every case the solutions of the 'reduced' (or 'resolvent') equation are 'functions' of the roots of the equation to be solved



Resolvents for cubics

For a cubic with roots x_1, x_2, x_3 there is a reduced equation whose roots are values of

$$y = \frac{1}{3}\left(x_1 + \alpha^2 x_2 + \alpha x_3\right)$$

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Lagrange identified this idea as a feature common to the methods for solving cubics presented by Cardano, Tschirnhaus, Bézout, and Euler

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Note 2: This theorem mutated several times, finally morphing into Lagrange's Theorem in group theory.

(See *Mathematics emerging*, §12.3.1, and also: Richard L. Roth, A History of Lagrange's Theorem on Groups, *Mathematics Magazine* **74**(2) (2001), pp. 99–108)

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Is there any hope of reducing it further?

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A long, confused and confusing account, which seems to have persuaded no-one except Italian pupils and colleagues?

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Proved his conjecture for $n = 6$

(See *Mathematics emerging*, §13.1.1.)

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The end of ‘classical algebra’ (?)