BO1 History of Mathematics Lecture X The 19th-century beginnings of 'modern algebra' Part 2: Groups

MT 2020 Week 5

▲□▶ ▲□▶ ▲三▶ ▲三▶ 三三 のへで

Évariste Galois (1811-1832)



Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

 submitted his ideas to the Academy in 1829, withdrew his articles January 1830 on Cauchy's advice

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

- submitted his ideas to the Academy in 1829, withdrew his articles January 1830 on Cauchy's advice
- resubmitted February 1830; lost after Fourier died in 1830

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

- submitted his ideas to the Academy in 1829, withdrew his articles January 1830 on Cauchy's advice
- resubmitted February 1830; lost after Fourier died in 1830
- resubmitted January 1831, rejected by Academy on Poisson's advice in July 1831

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

- submitted his ideas to the Academy in 1829, withdrew his articles January 1830 on Cauchy's advice
- resubmitted February 1830; lost after Fourier died in 1830
- resubmitted January 1831, rejected by Academy on Poisson's advice in July 1831

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

corrected by Galois up to his death by duel in 1832

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

- submitted his ideas to the Academy in 1829, withdrew his articles January 1830 on Cauchy's advice
- resubmitted February 1830; lost after Fourier died in 1830
- resubmitted January 1831, rejected by Academy on Poisson's advice in July 1831
- corrected by Galois up to his death by duel in 1832
- to be read in conjunction with Galois' Testamentary Letter of 29 May 1832 to Auguste Chevalier

Galois, in his writings 1829/30 (published 1846):

Galois, in his writings 1829/30 (published 1846):

 invented groups (of permutations) [note: Cauchy invented groups in 1845, almost certainly independently]

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Galois, in his writings 1829/30 (published 1846):

- invented groups (of permutations) [note: Cauchy invented groups in 1845, almost certainly independently]
- pre-invented fields (in 'Théorie des nombres' [published 1830] and as his 'rationally known quantities')

Galois, in his writings 1829/30 (published 1846):

- invented groups (of permutations) [note: Cauchy invented groups in 1845, almost certainly independently]
- pre-invented fields (in 'Théorie des nombres' [published 1830] and as his 'rationally known quantities')
- showed how to associate a group to a polynomial (its Galois group)

▲□▶ ▲□▶ ▲□▶ ▲□▶ ■ ● ●

Galois, in his writings 1829/30 (published 1846):

- invented groups (of permutations) [note: Cauchy invented groups in 1845, almost certainly independently]
- pre-invented fields (in 'Théorie des nombres' [published 1830] and as his 'rationally known quantities')
- showed how to associate a group to a polynomial (its Galois group)
- discovered a necessary and sufficient condition for solubility of an equation by radicals expressed in terms of the structure of its group

Galois, in his writings 1829/30 (published 1846):

- invented groups (of permutations) [note: Cauchy invented groups in 1845, almost certainly independently]
- pre-invented fields (in 'Théorie des nombres' [published 1830] and as his 'rationally known quantities')
- showed how to associate a group to a polynomial (its Galois group)
- discovered a necessary and sufficient condition for solubility of an equation by radicals expressed in terms of the structure of its group
- as an application, gave a necessary and sufficient condition for solubility of an irreducible equation of prime degree by radicals

Phintime. dat use consta Annie But a be to a anne. So por topico an age to parataking the littles a, byc, forther and to form of investigent for fair of the property devents . " on tale fortim to arises inversable par I plantition It Talimaellowent "more carines entre alles , mine our salts But la What remaining a badenit an saccord jor 2° supreguencest, que tate fastin des soumes ist. remall internalionant set convenieth for C. passes titiens to let the time (Some to car is Equilions algorized, a groups comment; sons soulors sin que de colour annungen est esperande se facteur alimentes les alfonstes a formitation possible and la ve beller, puisque ness a as , be forther symmitriques and suto a yout and afilm Bout to an telligistic that we , & l'a support - itait -South pop takalan Shot To which hat by a pringe the i to ha part four anding blifters it test artifring Hogad when good it i byit & faution particulier A day le armite feature & planning littles equitations at egal an agen in I quation , it min done and los in les quella in 7 pater goin with To an part guess to for the la second trant to fadion alignet la I'm substitutor the afree leave be outer.] Descrittation sully que sit l'épentes comis, ou when any other Star & burger a beause ine forder rationally V & win mote foquet by primitation at while a ge tate to social saint building not could and premie to program por consistions l'équation intéractifie sont } Same (Longer LII S 11) dint V plyt. a on when les ruins le Me Equales when and and my younger by caleddition any ben tates prover d'our win primets tim V P.V P.V Part aton up il a letter Course it Variant Engines & quitin n promotation provides in sensi Par V. litteriles granter That litter administray and Pm,V' The glosser for our and some to how sever by Pr.Y Fred V when hit station sull for int hapen on a los publitution Sil and medianis to subditation ST In effet, 1' but faction 5' & raines inversibile a a groupe , same it's is

Premier mémoire, dossier 1, folio 3 verso

Proposition I relates a given polynomial to a group of permutations (its Galois group)

Eleventh-hour marginal additions provide further explanation

イロト 人間 ト イヨト イヨト

The eleventh-hour marginal addition in translation:

The eleventh-hour marginal addition in translation:

Substitutions are the passage from one permutation to another.

The eleventh-hour marginal addition in translation:

Substitutions are the passage from one permutation to another.

The permutation from which one starts in order to indicate substitutions is completely arbitrary, ...

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

The eleventh-hour marginal addition in translation:

Substitutions are the passage from one permutation to another.

The permutation from which one starts in order to indicate substitutions is completely arbitrary, ...

... one must have the same substitutions, whichever permutation it is from which one starts. Therefore, if in such a group one has substitutions S and T, one is sure to have the substitution ST.

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

Évariste Galois, 29th May 1832, published 1846

(See *Mathematics emerging*, §13.1.2.)

Publication of Galois' results:



Publication of Galois' results:

1829-30: 5 articles inc. 'Sur la théorie des nombres'

▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ 三三 - のへぐ

Publication of Galois' results:

1829-30: 5 articles inc. 'Sur la théorie des nombres'

▲□▶ ▲□▶ ▲ □▶ ▲ □▶ □ のへぐ

1832: Testamentary Letter to Chevalier

Publication of Galois' results:

- 1829-30: 5 articles inc. 'Sur la théorie des nombres'
 - 1832: Testamentary Letter to Chevalier
 - 1846: letter and all other major papers by Liouville

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Publication of Galois' results:

- 1829-30: 5 articles inc. 'Sur la théorie des nombres'
 - 1832: Testamentary Letter to Chevalier
 - 1846: letter and all other major papers by Liouville

▲□▶ ▲□▶ ▲ □▶ ▲ □▶ □ のへぐ

1897: Liouville's edition re-published by Picard

Publication of Galois' results:

- 1829-30: 5 articles inc. 'Sur la théorie des nombres'
 - 1832: Testamentary Letter to Chevalier
 - 1846: letter and all other major papers by Liouville

- 1897: Liouville's edition re-published by Picard
- 1906/07/08: minor manuscripts published by Tannery

Publication of Galois' results:

- 1829-30: 5 articles inc. 'Sur la théorie des nombres'
 - 1832: Testamentary Letter to Chevalier
 - 1846: letter and all other major papers by Liouville

- 1897: Liouville's edition re-published by Picard
- 1906/07/08: minor manuscripts published by Tannery
 - 1962: complete Bourgne & Azra edition

Publication of Galois' results:

- 1829-30: 5 articles inc. 'Sur la théorie des nombres'
 - 1832: Testamentary Letter to Chevalier
 - 1846: letter and all other major papers by Liouville
 - 1897: Liouville's edition re-published by Picard
- 1906/07/08: minor manuscripts published by Tannery
 - 1962: complete Bourgne & Azra edition
- 25 Oct 2011: English/French bilingual edition by Peter Neumann

Galois in English (2011)

Heritage of European Mathematics

Peter M. Neumann

The mathematical writings of Évariste Galois

European Mathematical Society

▲□▶ ▲□▶ ▲三▶ ▲三▶ 三三 のへで

▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ 三三 - のへぐ

Meanwhile, in 1845 ...

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

and submitted a paper to the Paris Academy in March 1845;

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

and submitted a paper to the Paris Academy in March 1845;

in April 1845, Cauchy was given Bertrand's paper for review ...

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

and submitted a paper to the Paris Academy in March 1845;

in April 1845, Cauchy was given Bertrand's paper for review ...

and from September 1845 to January 1846 published a stream of papers on the same topic (and introducing a version of groups),

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

and submitted a paper to the Paris Academy in March 1845;

in April 1845, Cauchy was given Bertrand's paper for review ...

and from September 1845 to January 1846 published a stream of papers on the same topic (and introducing a version of groups),

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

giving his report on Bertrand's paper in November 1845,

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

and submitted a paper to the Paris Academy in March 1845;

in April 1845, Cauchy was given Bertrand's paper for review ...

and from September 1845 to January 1846 published a stream of papers on the same topic (and introducing a version of groups),

giving his report on Bertrand's paper in November 1845,

which was eventually published in November 1848

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

and submitted a paper to the Paris Academy in March 1845;

in April 1845, Cauchy was given Bertrand's paper for review ...

and from September 1845 to January 1846 published a stream of papers on the same topic (and introducing a version of groups),

giving his report on Bertrand's paper in November 1845,

which was eventually published in November 1848

(Peter M. Neumann, 'On the date of Cauchy's contributions to the founding of the theory of groups', *Bull. Austral. Math. Soc.* **40** (1989), 293–302.)

Cauchy's definition of a 'group' (1845):



Cauchy's definition of a 'group' (1845):

Consider substitutions $\binom{A}{B}$, $\binom{C}{D}$, $\binom{E}{F}$, ... and all those derived from them by multiplying them together one or more times in any order. These form a système de substitutions conjuguées (a system of conjoined substitutions).

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

Cauchy's definition of a 'group' (1845):

Consider substitutions $\binom{A}{B}$, $\binom{C}{D}$, $\binom{E}{F}$,... and all those derived from them by multiplying them together one or more times in any order. These form a système de substitutions conjuguées (a system of conjoined substitutions).

His purpose: for any function $f(x_1, x_2, ..., x_n)$ the substitutions that leave it unchanged (yielding 'valeurs égales') form such a system. The number of values of the function ('valeurs différentes') is the index of this system — that is $\frac{n!}{N}$, where N is the number of its members.

(日)((1))

Cauchy's definition of a 'group' (1845):

Consider substitutions $\binom{A}{B}$, $\binom{C}{D}$, $\binom{E}{F}$,... and all those derived from them by multiplying them together one or more times in any order. These form a système de substitutions conjuguées (a system of conjoined substitutions).

His purpose: for any function $f(x_1, x_2, ..., x_n)$ the substitutions that leave it unchanged (yielding 'valeurs égales') form such a system. The number of values of the function ('valeurs différentes') is the index of this system — that is $\frac{n!}{N}$, where N is the number of its members.

(日)((1))

Hence — a proof of his 1815 conjecture and more.

Académie des Sciences, Paris, *Grand Prix de Mathématiques*, 1860: subject announced April 1857 (Cauchy on committee, he died a month later):

What are the possibilities for the number of values of well defined functions containing a given number of letters, and how can one form the functions for which there exist a given number of values?

Three competitors:

- Émile Mathieu;
- Camille Jordan

(submitted their Paris doctoral dissertations);

Rev. Thomas Penyngton Kirkman

(submitted his essay 'The complete theory of groups').

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Three competitors:

- Émile Mathieu;
- Camille Jordan

(submitted their Paris doctoral dissertations);

Rev. Thomas Penyngton Kirkman

(submitted his essay 'The complete theory of groups').

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

None successful.

Three competitors:

- Émile Mathieu;
- Camille Jordan

(submitted their Paris doctoral dissertations);

Rev. Thomas Penyngton Kirkman

(submitted his essay 'The complete theory of groups'). None successful.

All interpreted the problem as 'find all subgroups of Sym(n)'.

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

Three competitors:

- Émile Mathieu;
- Camille Jordan

(submitted their Paris doctoral dissertations);

 Rev. Thomas Penyngton Kirkman (submitted his essay 'The complete theory of groups').
None successful.

All interpreted the problem as 'find all subgroups of Sym(n)'.

The competition stimulated:

- development of theory of (finite permutation) groups;
- merger of Galois' and Cauchy's independent theories.