BO1 History of Mathematics
Lecture X
The 19th-century beginnings of 'modern algebra'
Part 2: Groups

MT 2020 Week 5

Évariste Galois (1811–1832)

Galois and his groups (1)

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- to be read in conjunction with Galois' Testamentary Letter of 29 May 1832 to Auguste Chevalier

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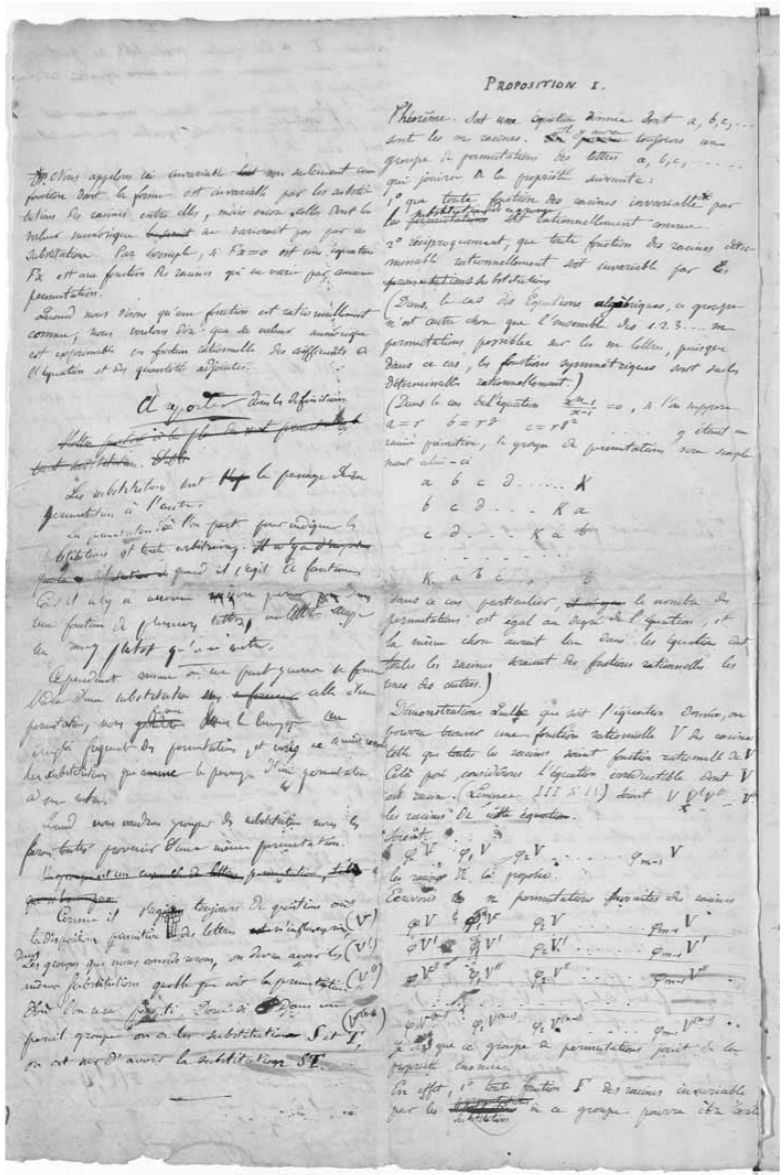
- as an application, gave a necessary and sufficient condition for solubility of an irreducible equation of prime degree by radicals

Galois and his groups (3)


PROPOSITION I.

Premier mémoire, dossier 1, folio 3 verso

Proposition I relates a given polynomial to a group of permutations (its Galois group)

Eleventh-hour marginal additions provide further explanation

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> *Substitutions are the passage from one permutation to another.*
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> *The permutation from which one starts in order to indicate substitutions is completely arbitrary, ...*
>
> *... one must have the same substitutions, whichever permutation it is from which one starts. Therefore, if in such a group one has substitutions S and T, one is sure to have the substitution ST.*

Évariste Galois, 29th May 1832, published 1846

(See *Mathematics emerging*, §13.1.2.)

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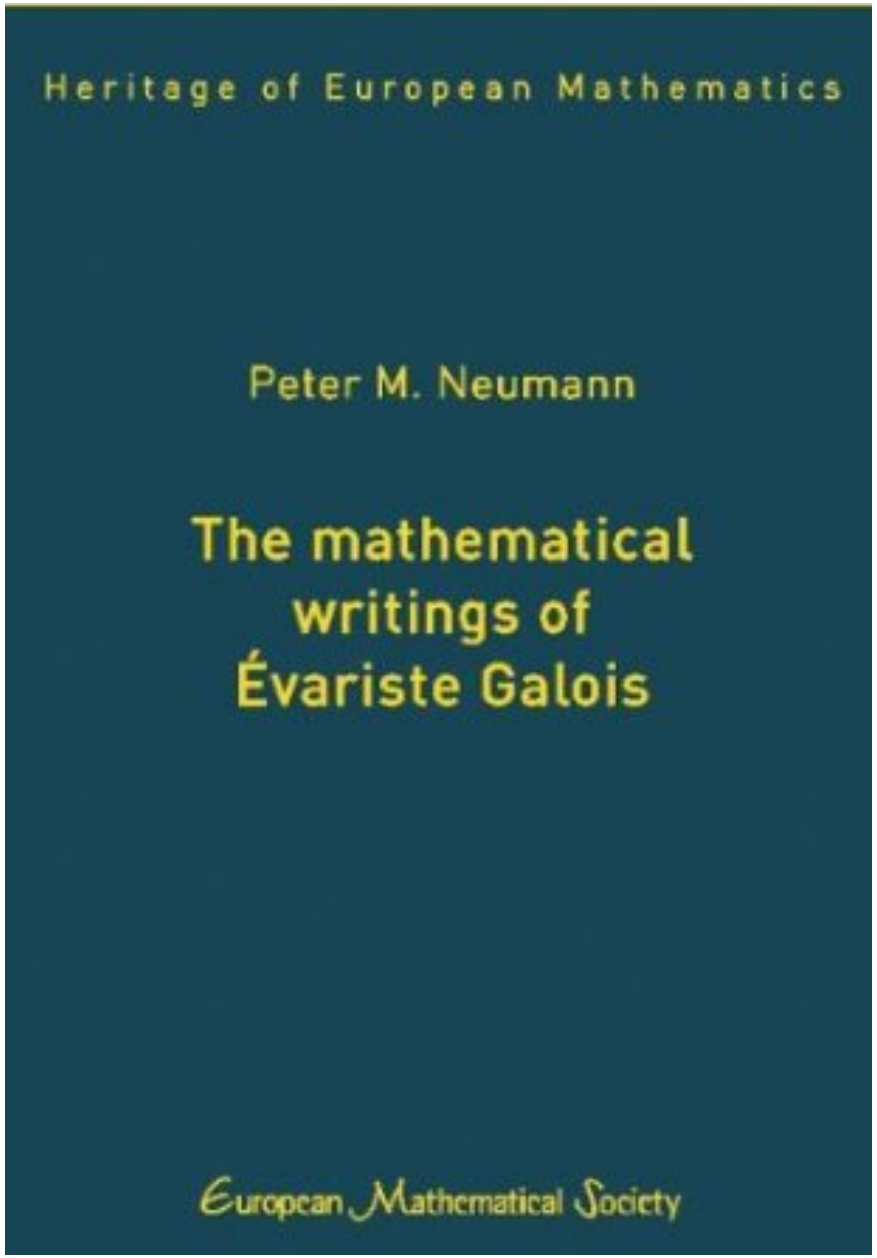
25 Oct 2011:	English/French bilingual edition by Peter Neumann

Galois in English (2011)



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(Peter M. Neumann, 'On the date of Cauchy's contributions to the founding of the theory of groups', *Bull. Austral. Math. Soc.* **40** (1989), 293–302.)

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Hence — a proof of his 1815 conjecture and more.

The Paris Grand Prix of 1860

Académie des Sciences, Paris, *Grand Prix de Mathématiques*, 1860: subject announced April 1857 (Cauchy on committee, he died a month later):

> *What are the possibilities for the number of values of well defined functions containing a given number of letters, and how can one form the functions for which there exist a given number of values?*

Grand Prix 1860: responses

Three competitors:

- Émile Mathieu;
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  (submitted their Paris doctoral dissertations);
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The competition stimulated:

- development of theory of (finite permutation) groups;
- merger of Galois' and Cauchy's independent theories.