# BO1 History of Mathematics
# Lecture X
# The 19th-century beginnings of ‘modern algebra’
# Part 3: The emergence of abstract algebra

MT 2020 Week 5

# Meanwhile, in Britain...

A sideline in the development of algebra in the 19th century was the notion of symbolical algebra that emerged in Britain c. 1830

# Meanwhile, in Britain...

A sideline in the development of algebra in the 19th century was the notion of symbolical algebra that emerged in Britain c. 1830

This was a response to an argument advanced by a (persistent) minority of British mathematicians that the notion of a negative number is invalid

# Meanwhile, in Britain...

A sideline in the development of algebra in the 19th century was the notion of symbolical algebra that emerged in Britain c. 1830

This was a response to an argument advanced by a (persistent) minority of British mathematicians that the notion of a negative number is invalid

It first appeared in print in the 1830 *A Treatise of Algebra* by George Peacock (1791–1858)

# Symbolical algebra

In symbolical algebra, symbols are regarded initially as being without interpretation, but may be manipulated according to specified rules; interpretation comes at the end of the process.

# Symbolical algebra

In symbolical algebra, symbols are regarded initially as being without interpretation, but may be manipulated according to specified rules; interpretation comes at the end of the process.

This approach was not accepted by all mathematicians in Britain, and sparked a debate about the nature of mathematical truth: can operation really precede interpretation?

# Symbolical algebra

In symbolical algebra, symbols are regarded initially as being without interpretation, but may be manipulated according to specified rules; interpretation comes at the end of the process.

This approach was not accepted by all mathematicians in Britain, and sparked a debate about the nature of mathematical truth: can operation really precede interpretation?

Contributors to symbolical algebra included George Peacock (1791–1858), Duncan Gregory (1813–1844), Augustus De Morgan (1806–1871), George Boole (1815–1864), and William Rowan Hamilton (1805–1865).

# Symbolical algebra

In symbolical algebra, symbols are regarded initially as being without interpretation, but may be manipulated according to specified rules; interpretation comes at the end of the process.

This approach was not accepted by all mathematicians in Britain, and sparked a debate about the nature of mathematical truth: can operation really precede interpretation?

Contributors to symbolical algebra included George Peacock (1791–1858), Duncan Gregory (1813–1844), Augustus De Morgan (1806–1871), George Boole (1815–1864), and William Rowan Hamilton (1805–1865).

But the idea of symbolical algebra had largely faded away by the middle of the century: one *could* work with arbitrary operations in an entirely abstract setting, but why would one want to?

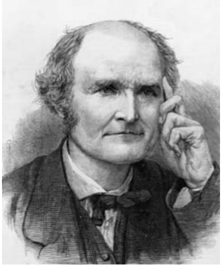
# Cayley and his groups (1)

Arthur Cayley, ‘On the theory of groups, as depending on the symbolic equation $\theta^n = 1$’ (1854):

# Cayley and his groups (1)

Arthur Cayley, 'On the theory of groups, as depending on the symbolic equation $\theta^n = 1$' (1854):

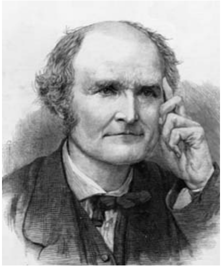
> *A set of symbols*
>
> $$1, \alpha, \beta, \ldots$$
>
> *all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a group.*



# Cayley and his groups (1)

Arthur Cayley, ‘On the theory of groups, as depending on the symbolic equation $\theta^n = 1$’ (1854):

> *A set of symbols*
>
> $$1, \alpha, \beta, \ldots$$
>
> *all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a group.*



Cayley widely attributed with introducing ‘abstract’ theory of groups

(See *Mathematics emerging*, §13.1.4.)

# Cayley and his groups (2)

Examples of groups of order 4:

- roots of $x^4 - 1 = 0$
- other examples from elliptic functions, quadratic forms
- matrices $(A, A^{-1}, A^T, (A^T)^{-1})$

# Cayley and his groups (2)

Examples of groups of order 4:

- roots of $x^4 - 1 = 0$
- other examples from elliptic functions, quadratic forms
- matrices $(A, A^{-1}, A^T, (A^T)^{-1})$

Examples of groups of order 6:

- permutations of three letters
- operations from his previous paper ‘properties of a caustic’

# Cayley and his groups (2)

Examples of groups of order 4:

- roots of $x^4 - 1 = 0$
- other examples from elliptic functions, quadratic forms
- matrices $(A, A^{-1}, A^T, (A^T)^{-1})$

Examples of groups of order 6:

- permutations of three letters
- operations from his previous paper 'properties of a caustic'

Key question for Cayley later: how many groups of order $n$?

# Cayley and his groups (2)

Examples of groups of order 4:

- roots of $x^4 - 1 = 0$
- other examples from elliptic functions, quadratic forms
- matrices $(A, A^{-1}, A^T, (A^T)^{-1})$

Examples of groups of order 6:

- permutations of three letters
- operations from his previous paper 'properties of a caustic'

Key question for Cayley later: how many groups of order $n$?

Cayley, 'On the theory of groups' (1878):

> *A group is defined by means of the laws of combinations of its symbols.*

# Weber's axioms, 1882

A System $G$ of $h$ elements of any kind, $\Theta_1, \Theta_2, \ldots, \Theta_h$ is called a <u>group of degree $h$</u>, if it satisfies the following conditions:

I. By some rule, which will be called composition or multiplication, from two elements of the system a new element of the system may be derived. In symbols:

$$\Theta_r \Theta_s = \Theta_t.$$

II. Always:

$$(\Theta_r \Theta_s)\Theta_t = \Theta_r(\Theta_s \Theta_t) = \Theta_r \Theta_s \Theta_t.$$

III. From $\Theta\Theta_r = \Theta\Theta_s$ and from $\Theta_r\Theta = \Theta_s\Theta$ follows $\Theta_r = \Theta_s$.

# Weber's axioms, 1882

A System $G$ of $h$ elements of any kind, $\Theta_1, \Theta_2, \ldots, \Theta_h$ is called a group of degree $h$, if it satisfies the following conditions:

I. By some rule, which will be called composition or multiplication, from two elements of the system a new element of the system may be derived. In symbols:

$$\Theta_r \Theta_s = \Theta_t.$$

II. Always:

$$(\Theta_r \Theta_s)\Theta_t = \Theta_r(\Theta_s \Theta_t) = \Theta_r \Theta_s \Theta_t.$$

III. From $\Theta\Theta_r = \Theta\Theta_s$ and from $\Theta_r\Theta = \Theta_s\Theta$ follows $\Theta_r = \Theta_s$.

Existence of identity and inverses appear as deductions from the axioms — incorporated as axioms by later authors

# On axiomatisation of groups

Peter M. Neumann, 'What groups were: a study of the development of the axiomatics of group theory', *Bull. Austral. Math. Soc.* **60** (1999), 285–301.

Christopher D. Hollings, '"Nobody could possibly misunderstand what a group is": a study in early twentieth-century group axiomatics', *Arch. Hist. Exact Sci.* **71**(5) (2017), 409–481.

# Rings and ideals

Ernst Kummer (1844):

- concerned with Fermat's last theorem, and quadratic forms
- worked with arithmetic of 'cyclotomic integers' $a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1}$ where $\theta$ is primitive $n$-th root of 1
- discovered that unique factorisation need not hold
- devised the concept of 'ideal' factors

# Rings and ideals

Ernst Kummer (1844):

- concerned with Fermat's last theorem, and quadratic forms
- worked with arithmetic of 'cyclotomic integers' $a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1}$ where $\theta$ is primitive $n$-th root of 1
- discovered that unique factorisation need not hold
- devised the concept of 'ideal' factors

Richard Dedekind, 'Sur la théorie des nombres entiers algébriques' (1877) and famous appendices to his editions of Dirichlet's *Lectures on Number Theory*:

- changed Kummer's 'ideal numbers' to 'ideals'
- worked also with rings of numbers [domains] and fields of numbers [Körper]

# ‘Abstract algebra’ begins to form

Specific instances of fields studied by Galois, Kronecker, Dedekind, and others. First axiomatic definition due to Weber, 1893.

# ‘Abstract algebra’ begins to form

Specific instances of fields studied by Galois, Kronecker, Dedekind, and others. First axiomatic definition due to Weber, 1893.

Ernst Steinitz, ‘Algebraische Theorie der Körper’, 1910: first comprehensive presentation of the theories of fields, modules, and vector spaces [to be revisited in a later lecture]

# 'Abstract algebra' begins to form

Specific instances of fields studied by Galois, Kronecker, Dedekind, and others. First axiomatic definition due to Weber, 1893.

Ernst Steinitz, 'Algebraische Theorie der Körper', 1910: first comprehensive presentation of the theories of fields, modules, and vector spaces [to be revisited in a later lecture]

Specific rings studied by Dedekind, Hilbert, ... Early axiomatic definition given by Fraenkel in 1914: not quite the same as the modern definition.

# ‘Abstract algebra’ begins to form

Specific instances of fields studied by Galois, Kronecker, Dedekind, and others. First axiomatic definition due to Weber, 1893.

Ernst Steinitz, ‘Algebraische Theorie der Körper’, 1910: first comprehensive presentation of the theories of fields, modules, and vector spaces [to be revisited in a later lecture]

Specific rings studied by Dedekind, Hilbert, ... Early axiomatic definition given by Fraenkel in 1914: not quite the same as the modern definition.

Abstract algebra given an early boost (in the USA) via a short-lived obsession with ‘postulate analysis’: the study of systems of axioms for their own sake

# ‘Abstract algebra’ takes off

Comprehensive abstract study of (commutative) rings initiated by Emmy Noether in the 1920s, sometimes mirroring the earlier ‘concrete’ work of Dedekind: ‘Es steht alles schon bei Dedekind’.

# ‘Abstract algebra’ takes off

Comprehensive abstract study of (commutative) rings initiated by Emmy Noether in the 1920s, sometimes mirroring the earlier ‘concrete’ work of Dedekind: ‘Es steht alles schon bei Dedekind’.

Noether’s lectures (in Göttingen) united with those of Emil Artin (Hamburg) in B. L. van der Waerden’s highly influential textbook *Moderne Algebra* (1930).

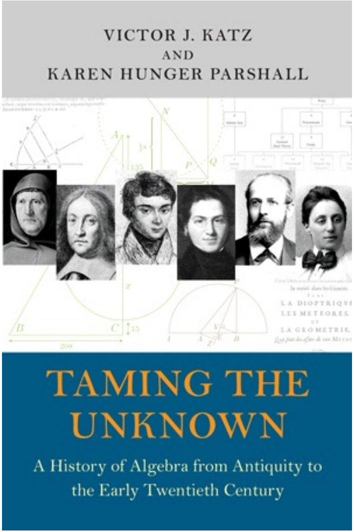
# ‘Abstract algebra’ takes off

Comprehensive abstract study of (commutative) rings initiated by Emmy Noether in the 1920s, sometimes mirroring the earlier ‘concrete’ work of Dedekind: ‘Es steht alles schon bei Dedekind’.

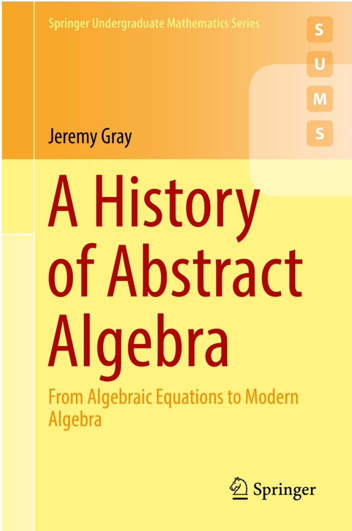
Noether’s lectures (in Göttingen) united with those of Emil Artin (Hamburg) in B. L. van der Waerden’s highly influential textbook *Moderne Algebra* (1930).

Abstract point of view now dominant, with many different objects studied: groups, fields, rings, integral domains, semigroups, algebras, lattices, semirings, quasigroups, ...

# Overviews of the topics of lectures IX and X




(Princeton Univ. Press, 2014)




(Springer, 2018)