BO1 History of Mathematics
Lecture XI
19th-century rigour in real analysis
Part 1: Uniformity

MT 2020 Week 6

Summary

Part 1

- New difficulties emerge
- Continuity and convergence

Part 2

- Integration
- The Fundamental Theorem of Calculus
- New ideas about integration

Recall from lecture VIII: Fourier series, 1822

Joseph Fourier, *Théorie analytique de la chaleur* [*Analytic theory of heat*] (1822):

Suppose that $\phi(x) = a \sin x + b \sin 2x + c \sin 3x + \cdots$

and also that $\phi(x) = x\phi'(0) + \frac{1}{6}x^3\phi'''(0) + \cdots$

After many pages of calculations, multiplying and comparing power series, Fourier found that the coefficient of $\sin nx$ must be

$$\frac{2}{\pi}\int_0^\pi \phi(x) \sin nx \, dx$$

Fourier's derivation was based on 'naive' manipulations of infinite series. It was ingenious but non-rigorous, shaky.

BUT it led to profound results

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- functions — what exactly should they be?
- convergence — what exactly should it be?
- convergence of functions — what properties are preserved?
- integration — what exactly should it be?
- existence of limits — what are the essential properties of real numbers? [Lecture XII]

Recall from Lecture VIII: Cauchy sequences, 1821

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0,\ u_1,\ u_2, \ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit* s*: in other words, it is necessary and sufficient that for infinitely large values of the number* n*, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit* s*, and consequently from each other, by infinitely small quantities.*

Cauchy and continuity revisited

In *Cours d'analyse*, p. 34, Cauchy defined a function f to be continuous between certain limits if, for each x between those limits, the value of $f(x)$ is unique and finite, and $|f(x+\alpha) - f(\alpha)|$, where α is indefinitely small, decreases indefinitely with α.

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So Cauchy defined continuity on an interval, rather than at a point.

He went on to derive basic results concerning continuous functions: that the composition of two continuous functions is continuous, the Intermediate Value Theorem, etc.

A theorem of Cauchy (1821)

Cauchy, *Cours d'analyse*, pp. 131–132:

> *When the various terms of a series are functions of a variable x, continuous with respect to this variable in the neighbourhood of a particular value for which the series is convergent, the sum s of the series is also, in the neighbourhood of this value, a continuous function of x.*

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Not true!

Cauchy's argument

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NB. All notation except '$\sum$' is Cauchy's.

Cauchy's argument

est convergente, la somme de cette série est représentée par

$$u_0 + u_1 + u_2 + u_3 + \&c.\ldots$$

En vertu de cette convention, la valeur du nombre e se trouvera déterminée par l'équation

$$(6) \quad e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.\ldots;$$

et, si l'on considère la progression géométrique

$$1,\ x,\ x^2,\ x^3,\ \&c.\ldots,$$

on aura, pour des valeurs numériques de x inférieures à l'unité,

$$(7) \quad 1 + x + x^2 + x^3 + \&c.\ldots = \frac{1}{1-x}.$$

La série

$$u_0,\ u_1,\ u_2,\ u_3,\ \&c.\ldots$$

étant supposée convergente, si l'on désigne sa somme par s, et par s_n la somme de ses n premiers termes, on trouvera

$$s = u_0 + u_1 + u_2 + \ldots + u_{n-1} + u_n + u_{n+1} + \&c.\ldots$$
$$= s_n + u_n + u_{n+1} + \&c.\ldots,$$

et par suite

$$s - s_n = u_n + u_{n+1} + \&c.\ldots$$

De cette dernière équation il résulte que les quantités

$$u_n,\ u_{n+1},\ u_{n+2},\ \&c.\ldots$$

formeront une nouvelle série convergente dont la somme sera équivalente à $s - s_n$. Si l'on représente cette même somme par r_n, on aura

$$s = s_n + r_n;$$

et r_n sera ce qu'on appelle le *reste* de la série (1) à partir du n.me terme.

Lorsque, les termes de la série (1) renfermant une même variable x, cette série est convergente, et ses différens termes fonctions continues de x, dans le voisinage d'une valeur particulière attribuée à cette variable;

$$s_n,\quad r_n \quad \text{et} \quad s$$

sont encore trois fonctions de la variable x, dont la première est évidemment continue par rapport à x dans le voisinage de la valeur particulière dont il s'agit. Cela posé, considérons les accroissemens que reçoivent ces trois fonctions, lorsqu'on fait croître x d'une quantité infiniment petite α. L'accroissement de s_n sera, pour toutes les valeurs possibles de n, une quantité infiniment petite; et celui de r_n deviendra insensible en même temps que r_n, si l'on attribue à n une valeur très-considérable. Par suite, l'accroissement de la fonction s ne pourra être qu'une quantité infiniment petite. De cette remarque on déduit immédiatement la proposition suivante.

1.er THÉORÈME. *Lorsque les différens termes de la série (1) sont des fonctions d'une même variable x,*

A modern counterexample

For each $n \in \mathbb{N}$, define continuous functions f_n by

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n}; \\ nx & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}; \\ +1 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

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But we see that $s_n \to s$ as $n \to \infty$, where

$$s(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ +1 & \text{if } x > 0, \end{cases}$$

which is discontinuous at $x = 0$.

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$$r_n(x) = \begin{cases} -1 - nx & \text{if } -\frac{1}{n} \leq x < 0; \\ 0 & \text{if } x = 0; \\ 1 - nx & \text{if } 0 < x \leq \frac{1}{n}. \end{cases}$$

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For each x, $r_n(x) \to 0$ as $n \to \infty$, but this does not happen simultaneously for all values of x.

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Cauchy clearly didn't make this distinction — but should this really be regarded as a 'mistake'?

Reactions to Cauchy's 'mistake'

Abel (Crelle's Journal, 1826) on the *Cours d'analyse*:

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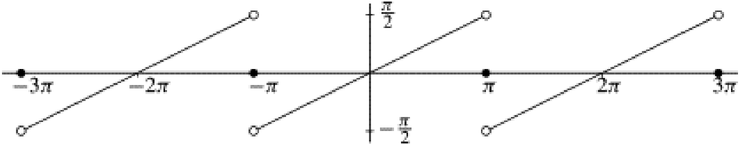
Four pages later, on Cauchy's 'theorem' on sums of continuous functions:

> *it seems to me that the theorem admits exceptions. For example, the series*
>
> $$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$
>
> *is discontinuous for every value* $(2m+1)\pi$ *of* x*,* m *being a whole number. There are, as one knows, many series of this kind.*

Abel's counterexample

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$



Abel's counterexample elsewhere

Abel to Holmboe, January 1826:

> *One applies all operations to infinite series as if they were finite, but is this permissible? I think not. — Where is it proved that one gets the differential of an infinite series by differentiating each term? It is easy to give an example for which this is not true, e.g.*
>
> $$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots .$$
>
> *Differentiation gives*
>
> $$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \cdots \text{etc.}$$
>
> *a result which is quite false because this series is divergent.*

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(See: Henrik Kragh Sørensen, Exceptions and counterexamples: Understanding Abel's comment on *Cauchy's Theorem*, *Historia Mathematica* **32** (2005) 453–480)

Abel's interpretation

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Similarly, what did Abel mean by 'continuity' and 'convergence'? The same as Cauchy? Or did he use a similar form of words but with a different meaning?

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- Karl Weierstrass (lectures in Berlin), 1841;

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See: G. H. Hardy, 'Sir George Stokes and the concept of uniform convergence', *Proc. Camb. Phil. Soc.* **19** (1918) 148–156 (also: *Collected Papers of G. H. Hardy*, vol. VII, 505–513)

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Theorem. If the different terms of the series

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are functions of a real variable x, continuous with respect to this variable within the given limits; and if, in addition, the sum

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always becomes infinitely small for infinitely large values of the whole numbers n and $n' > n$, then the series will be convergent and the sum of the series will be, within the given limits, a continuous function of the variable x.

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But it was becoming clear that the language of infinities and infinitesimals was inadequate for expressing the problems at hand.