BO1 History of Mathematics
Lecture XII
19th-century rigour in real analysis, continued
Part 1: Completeness

MT 2020 Week 6

Summary

Part 1

- Proofs of the Intermediate Value Theorem revisited
- Convergence and completeness

Part 2

- Dedekind and the continuum

Part 3

- Cantor and numbers and sets
- Where and when did sets emerge?
- Early set theory
- Set theory as a language

The Intermediate Value Theorem (1)

Bolzano's criticisms (1817) of existing proofs:

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But Bolzano assumed the existence of the limit.

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Cauchy's 2nd proof in a different context (p. 460): a numerical method for finding roots of equations — tacitly assumes that bounded monotone sequences of real numbers converge [see Lecture VIII].

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  3. *A monotonic bounded sequence converges to a limit (taken for granted by Cauchy in 1821).*

(*Mathematics emerging*, §16.3.1.)

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All equivalent

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All of the above relies upon an intuitive notion of real number — so perhaps provide a formal definition of these? One that includes the idea of completeness?