BO1 History of Mathematics Lecture XII 19th-century rigour in real analysis, continued Part 2: Real numbers

MT 2020 Week 6

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Richard Dedekind (1831–1916)



Stetigkeit und . irrationale <u>**3ahlen**</u>. Richard Dedekind. Professor ber höheren Mathematif am Collegium Carolinum zu Braunichmela Braunfcweig, Drud und Berlag von Friedrich Bieweg und Sohn. 1872.

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Response eventually published in *Stetigkeit und irrationale Zahlen* (1872) [translated as *Continuity and irrational numbers* by Wooster Woodruff Beman, 1901]

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I find the essence of continuity in the converse, i.e., in the following principle:

"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions." But Dedekind couldn't *prove* this property, so he had to take it as an axiom:

The assumption of this property for the line is nothing but an Axiom, through which alone we attribute continuity to the line, through which we understand continuity in the line.

(See *Mathematics emerging*, §16.3.2.)

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Hence Dedekind cuts (or sections, from the original German Schnitt).

Start from the system of rational numbers *R* (assumed known)

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Separate R into two classes A₁ and A₂ such that
for any a₁ in A₁, a₁ < a₂ for every a₂ in A₂
for any a₂ in A₂, a₂ > a₁ for every a₁ in A₁

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Whenever, then, we have to do with a cut produced by no rational number, we create a new irrational number, which we regard as completely defined by this cut ...

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β ligen. 39: β < α, [n iệt α < α; mitặn gebiet α ber Gölift A, und jaigith and der Gölift \mathbb{K}_1 an, und be ugaith β < cif.] o gobiet and β beriftlen Gölift \mathbb{K}_1 an, nei joe 3,00 in \mathbb{K}_1 größer ift als jele 3,60 i m \mathbb{K}_1 . 38 aber β > α, [n ift α > 3, mith gebiet α < C Gölift A, und bight and perform Gölift \mathbb{K}_2 an, med jele 3,60 i m \mathbb{K}_1 , be gebiet and β beriftlen Gölift \mathbb{K}_2 an, med jele 3,60 i m \mathbb{K}_1 for gebiet and β beriftlen Gölift \mathbb{K}_2 are gobiet jele son a triftleture ift als jele 3,61 i m \mathbb{K}_2 . Nithin gobiet jele son a criftleture ift als jele Gölift i i m $(b = 1)^{-1}$ \mathbb{K}_2 an i, i nadöben β < a, ober β > a ii; jelgift ift i end i i \mathbb{K}_2 . In a tilt eine min öffender bie einige 3,64, und melde bie 3 getigung. son \mathbb{R} in bie Gölift \mathbb{K}_2 , \mathbb{K}_2 berougedradit init. West su be notift next.

§. 6.

Rechnungen mit reellen Bablen.

Um ingend eine Rechnung mit zwei rerfelt Jahler n. β auf ich Rechnungen mit teinischer Jahler aufrächglichen. Kommt einur barauf, auf ben Schnitten (A_i , A_i) umb (B_i , B_i), undöge band der Sahlern a umb β im Schlumen B herroragefracht inrechen. Der Schnitt (D_i) zu behintern, zweicher bem Rechnungsteiluhter γ entityrochen foll. 3ch örlegknitte mich hier auf bie Durchführung bes einschlußen Beichliche, ben Röhnen.

ℜit e transb eine rationale 3ρδ(jo nežme man fit in bie Glöffe G, auf, neura ei eine 3ρδ(a), in *A*, un bie eine 3ρδ(b), in *B*, nen ber äftt gleich, haß ihre Gumme a₁ + b₁ ≥ c wirte; alle anberen rationaler 3ρδ(ein c netpen man in bie Glaffe (C, anf. Delef Gling, aller cutanteri 3ρδ(ein in bie behen Glaffen (C, *G*, bibbet öffenbar einer Gehnitt, meil jebe 3ρδ(a), in *G*, lieftner išt all sip Sabla (a) in *G*. Einharm tobel 2, solden a, *A* rational(, 16 ii) frèc Dedekind showed how to add two cuts, and how to use them in limiting arguments — but did little else with them.

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Significance: a major step towards

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Significance: a major step towards

- understanding completeness, and
- giving a rigorous definition of an irrational number, hence
- setting the foundations of analysis onto a sound logical basis.

Stetigkeit und irrationale Zahlen reprinted many times, often in conjunction with the later essay Was sind und was sollen die Zahlen? (1888) [see below].

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A good modern (historically sensitive) account can be found in: Leo Corry, *A brief history of numbers*, OUP, 2015, §10.6.

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Later constructions by many mathematicians and philosophers — such as

- Carl Johannes Thomae, 1880, 1890;
- ▶ Giuseppe Peano, 1889, 1891;
- Gottlob Frege, 1884, 1893, 1903;
- Otto Hölder, 1901;

CARDINAL ARITHMETIC [PART III *110.632. \vdash : $\mu \in \text{NC}$, \supset , $\mu + 1 = \hat{E}[(\pi v), v \in E, E - t'v \in \text{sm}^{\prime\prime}\mu]$ Dem. F. #110:631 . #51-911-99 . 7 $\vdash : \operatorname{Hp} \cdot \mathfrak{I} , \mu + \iota 1 = \hat{\xi} \{ (\mathfrak{g} \gamma, y) , \gamma \in \operatorname{sm}^{\prime \prime} \mu , y \in \xi , \gamma = \xi - \iota^{\prime} y \}$ [*13·195] $= \hat{\xi} \{(\Im y), y \in \xi, \xi - \iota' y \in \operatorname{sm}^{\iota} \mu\} : \supset \vdash$. Prop *110.64. F.0+.0=0 F#110:621 *110:641, +, 1+, 0=0+, 1=1 [*110:51:61, *101:2] *110.642. +, 2+, 0 = 0+, 2 = 2 [*110.51.61, *101.31] *110.643. ⊢ , 1 +, 1 = 2 Dem. F. *110:632 . *101:21:28 . D $\vdash .1 + 1 = \hat{\xi}[(\pi y) \cdot y \cdot \xi \cdot \xi - \iota' y \cdot 1]$ [#54:3] = 2. **>** F. Prop The above proposition is occasionally useful. It is used at least three times, in \$113.66 and \$120.123.472. \$110771 are required for proving \$11072, and \$11072 is used in #117.3, which is a fundamental proposition in the theory of greater and less. *1107. \vdash : $\beta \subset \alpha$, \supset , $(\forall \mu)$, $\mu \in NC$, $Nc'\alpha = Nc'\beta + \mu$ Dem. $\vdash . \ast 24 \cdot 411 \cdot 21 \cdot \supset \vdash : Hp \cdot \supset . \alpha = \beta \cup (\alpha - \beta) \cdot \beta \cap (\alpha - \beta) = \Lambda .$ [*110.32] \supset . Ne' α = Ne' β + Ne' $(\alpha - \beta)$: \supset +. Prop *11071. $\vdash : (\Im \mu)$. Ne' $\alpha = \operatorname{Ne'}\beta +_{\alpha} \mu \cdot \mathcal{I} \cdot (\Im \delta) \cdot \delta \operatorname{sm} \beta \cdot \delta \mathcal{C} \alpha$ Dem +.*1003.*1104.> $\vdash : Nc^{t} \alpha = Nc^{t} \beta +_{c} \mu \cdot \Im \cdot \mu e NC - \iota^{t} \Lambda$ (1) $\vdash . *110^{\cdot}3 \cdot \supset \vdash : \operatorname{Ne}^{t} \alpha = \operatorname{Ne}^{t} \beta + \operatorname{e} \operatorname{Ne}^{t} \gamma \cdot \equiv \cdot \operatorname{Ne}^{t} \alpha = \operatorname{Ne}^{t} (\beta + \gamma) \cdot$ [#100·3·31] $\Im, \alpha \operatorname{sm}(\beta + \gamma)$. [*73.1] (πR) , $R \in 1 \rightarrow 1$, $D^{i}R = \alpha$, $\Pi^{i}R = \perp \Lambda_{\gamma}^{\prime \prime} \iota^{\prime \prime}\beta \lor \Lambda_{\delta} \perp^{\prime \prime} \iota^{\prime \prime}\gamma$, [*37.15] \Im . $(\Im R)$, $R \in 1 \rightarrow 1$, $\downarrow \Lambda$, " ι " $\beta \subset (\Box R, R" \downarrow \Lambda$, " ι " $\beta \subset \alpha$. [#110.12.*73.22] **Ο**. (ηδ).δ**C**α.δ sm β (2)F.(1).(2). ⊃F. Prop

Alfred North Whitehead and Bertrand Russell, *Principia mathematica*, 3 vols., Cambridge University Press, 1910, 1912, 1913

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CARDINAL ARITHMETIC [PART III *110.632. \vdash : $\mu \in \text{NC}$, \supset , $\mu + 1 = \hat{E}[(\pi v), v \in E, E - t'v \in \text{sm}^{\prime\prime}\mu]$ Dem. F. #110:631 . #51-911-99 . 7 $\vdash : \operatorname{Hp} \cdot \mathfrak{I} , \mu + \iota 1 = \hat{\xi} \{ (\mathfrak{g} \gamma, y) , \gamma \in \operatorname{sm}^{\prime \prime} \mu , y \in \xi , \gamma = \xi - \iota^{\prime} y \}$ [*13·195] $= \hat{\xi} \{(\Im y), y \in \xi, \xi - \iota' y \in \operatorname{sm}^{\iota} \mu\} : \supset \vdash$. Prop *110.64. F.0+.0=0 F#110:621 *110:641, +, 1+, 0=0+, 1=1 [*110:51:61, *101:2] *110.642. +, 2+, 0 = 0+, 2 = 2 [*110.51.61, *101.31] *110.643. ⊢ , 1 +, 1 = 2 Dem. F. *110:632 . *101:21:28 . D $\vdash .1 + 1 = \hat{\xi}[(\pi y) \cdot y \cdot \xi \cdot \xi - \iota' y \cdot 1]$ [#54:3] = 2. **>** F. Prop The above proposition is occasionally useful. It is used at least three times, in \$113.66 and \$120.123.472. \$110771 are required for proving \$11072, and \$11072 is used in #117.3, which is a fundamental proposition in the theory of greater and less. *1107. \vdash : $\beta \subset \alpha$, \supset , $(\forall \mu)$, $\mu \in NC$, $Nc'\alpha = Nc'\beta + \mu$ Dem. $\vdash . \ast 24 \cdot 411 \cdot 21 \cdot \supset \vdash : Hp \cdot \supset . \alpha = \beta \cup (\alpha - \beta) \cdot \beta \cap (\alpha - \beta) = \Lambda .$ [*110.32] \supset . Ne' α = Ne' β +_e Ne' $(\alpha - \beta)$: \supset \vdash . Prop *11071. $\vdash : (\Im \mu)$. Ne' $\alpha = \operatorname{Ne'}\beta +_{\alpha} \mu \cdot \mathcal{I} \cdot (\Im \delta) \cdot \delta \operatorname{sm} \beta \cdot \delta \mathcal{C} \alpha$ Dem F.*1003.*1104. > $\vdash : Nc^{t} \alpha = Nc^{t} \beta +_{c} \mu \cdot \Im \cdot \mu e NC - \iota^{t} \Lambda$ (1) $\vdash . *110^{\cdot}3 \cdot \supset \vdash : \operatorname{Ne}^{t} \alpha = \operatorname{Ne}^{t} \beta + \operatorname{e} \operatorname{Ne}^{t} \gamma \cdot \equiv \cdot \operatorname{Ne}^{t} \alpha = \operatorname{Ne}^{t} (\beta + \gamma) \cdot$ [#100:3:31] $\Im, \alpha \operatorname{sm}(\beta + \gamma)$. [*73.1] (πR) , $R \in 1 \rightarrow 1$, $D^{i}R = \alpha$, $\Pi^{i}R = \perp \Lambda_{\gamma}^{\prime \prime} \iota^{\prime \prime}\beta \lor \Lambda_{\delta} \perp^{\prime \prime} \iota^{\prime \prime}\gamma$, [*37.15] \Im . $(\Im R)$, $R \in 1 \rightarrow 1$, $\downarrow \Lambda$, " ι " $\beta \subset (\Box R, R" \downarrow \Lambda$, " ι " $\beta \subset \alpha$. [#11012.#7322] **Ο**. (98).δ **C**α.δ sm / (2)F.(1).(2). ⊃F. Prop

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NB. This is not the source of our axioms for the reals.

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