BO1 History of Mathematics
Lecture XII
19th-century rigour in real analysis, continued
Part 3: Sets

MT 2020 Week 6

New ideas

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This is by no means an exhaustive list of examples; see *Mathematics emerging*, §18.2 for others.

Formalisation of the concept of a set

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> *By an "aggregate" (Menge) we are to understand any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects m of our intuition or our thought.*

Cantor and the continuum

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Cantor's first great insight regarding sets (1873): infinite sets can have different sizes.

Cantor's first proof that the continuum is uncountable

Proposition: Given any sequence of real numbers $\omega_1, \omega_2, \omega_3, \ldots$ and any interval $[\alpha, \beta]$, there is a real number in $[\alpha, \beta]$ that is not contained in the given sequence.

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Proof proceeds by construction of a sequence of nested intervals $[\alpha, \beta] \supseteq [\alpha_1, \beta_1] \supseteq [\alpha_2, \beta_2] \supseteq [\alpha_3, \beta_3] \supseteq \cdots$. Cantor considered the different cases where the sequence terminates or does not, but in all instances he constructed a real number in the interval that does not lie in the original sequence.

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Next suppose that the continuum is countable, i.e., that the real numbers may be listed $\omega_1, \omega_2, \omega_3, \ldots$. But then there is a real number in any interval $[\alpha, \beta]$ that does not belong to this list — a contradiction.

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The more famous diagonal argument came later (1891).

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Cantor to Dedekind (1877): there is a one-to-one correspondence between a line and the plane — "Je le vois, mais je ne le crois pas!" ("I see it, but I don't believe it!")

Cantor's *Mengenlehre*

Developed at the end of the nineteenth century (1878–1897): a general theory of sets and of transfinite numbers — infinite cardinals (e.g., $\#\mathbb{N} = \aleph_0$, $\#\mathbb{R} = c$), transfinite ordinals, ...

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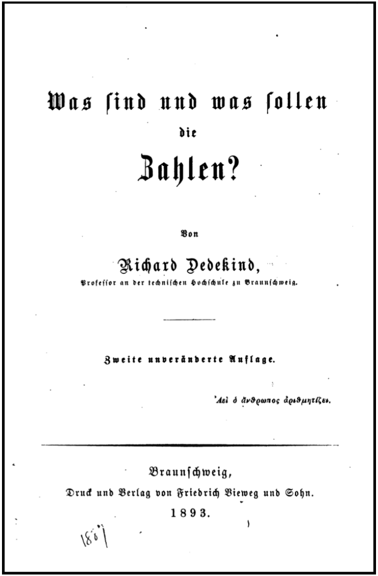
Was sind und was sollen die Zahlen?



Was sind und was sollen
die
Zahlen?

Von

Richard Dedekind,
Professor an der technischen Hochschule zu Braunschweig.

Zweite unveränderte Auflage.

Ἀεὶ ὁ ἄνθρωπος ἀριθμητίζει.

Braunschweig,
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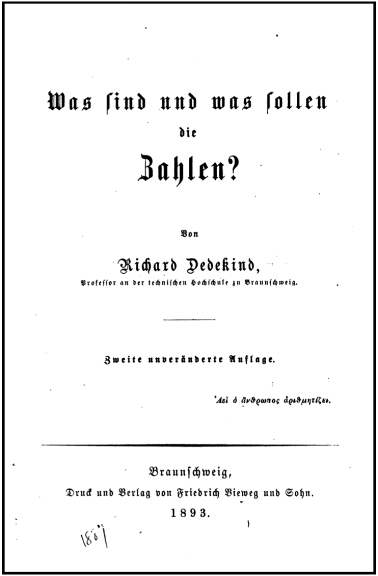

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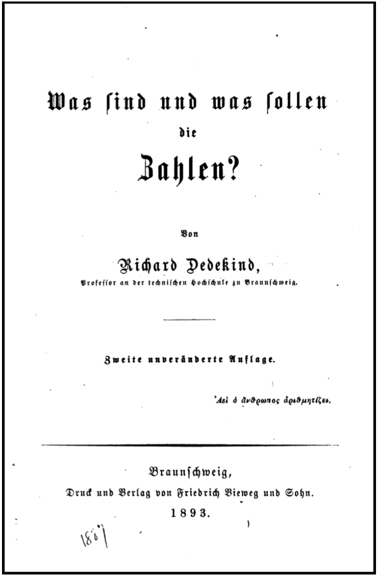
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- an axiomatisation of the natural numbers (soon simplified by Peano).

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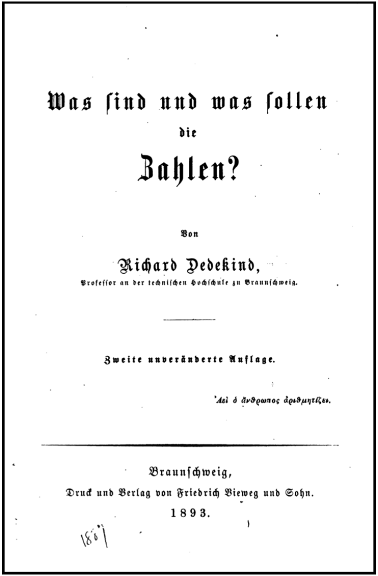
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Also includes a definition of a function as a mapping between sets (p. 6):

“By a mapping of a system S we understand a law according to which every determinate element s of S is associated with a determinate thing which is called the *image* of s and is denoted by $\phi(s)$ …”

Was sind und was sollen die Zahlen?

Extract from William Ewald, *From Kant to Hilbert: a source book in the foundations of mathematics*, OUP, 1996, vol. II, p. 790:

> *The title of Dedekind's paper is subtle: rigidly translated it asks 'What are, and what ought to be, the numbers?' But sollen here carries several senses—among them, 'What is the best way to regard the numbers?'; 'What is the function of numbers?; 'What are numbers supposed to be?'. But perhaps Dedekind's title is famous enough to be left in the original.*

W. W. Beman translated the essay under the title *The nature and meaning of numbers* (1901).

Was sind und was sollen die Zahlen?

(wegen der Aehnlichkeit von φ) auch a' und jedes Element u' verschieden von a und folglich in T enthalten sein; mithin ist $\psi(T) 3 T$, und da T endlich ist, so muß $\psi(T) = T$, also $\mathfrak{M}(a', U') = T$ sein. Hieraus folgt aber (nach 15)

$$\mathfrak{M}(a', a, U') = \mathfrak{M}(a, T),$$

d. h. nach dem Obigen $S' = S$. Also ist auch in diesem Falle der erforderliche Beweis geführt.

§. 6.

Einfach unendliche Systeme. Reihe der natürlichen Zahlen.

71. Erklärung. Ein System N heißt einfach unendlich, wenn es eine solche ähnliche Abbildung φ von N in sich selbst giebt, daß N als Kette (44) eines Elementes erscheint, welches nicht in $\varphi(N)$ enthalten ist. Wir nennen dies Element, das wir im Folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von N und sagen zugleich, das einfach unendliche System N sei durch diese Abbildung φ geordnet. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§. 4), so besteht mithin das Wesen eines einfach unendlichen Systems N in der Existenz einer Abbildung φ von N und eines Elementes 1, die den folgenden Bedingungen $\alpha, \beta, \gamma, \delta$ genügen:

α. $N' 3 N$.

β. $N = 1_0$.

γ. Das Element 1 ist nicht in N' enthalten.

δ. Die Abbildung φ ist ähnlich.

Offenbar folgt aus α, γ, δ, daß jedes einfach unendliche System N wirklich ein unendliches System ist (64), weil es einem echten Theile N' seiner selbst ähnlich ist.

72. Satz. In jedem unendlichen Systeme S ist ein einfach unendliches System N als Theil enthalten.

Written in an explicitly set-theoretic language

(But with slightly different notation from ours.)

For a summary, see: Kathryn Edwards, 'Richard Dedekind (1831–1916)', *Mathematics Today* **52**(1) (Feb 2016) 212–215

Set theory in our lives

Set theory as an effective language for mathematics:

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Set theory as an effective language for mathematics:

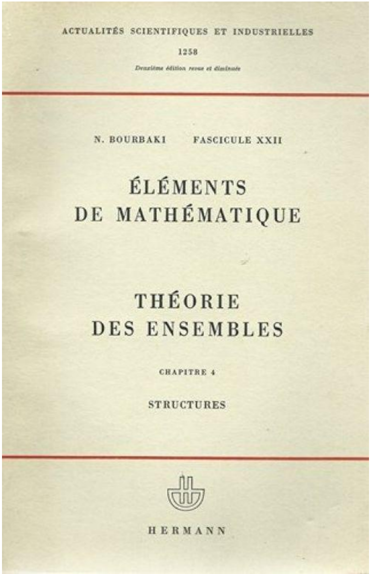
- Set-builder notation

Set theory in our lives

Set theory as an effective language for mathematics:

- Set-builder notation
- Unification of ideas concerning functions and relations

Nicolas Bourbaki (1934–????)



Collective of French mathematicians who set out to reformulate mathematics on extremely formal, abstract, structural lines — the language of sets has a significant role to play.

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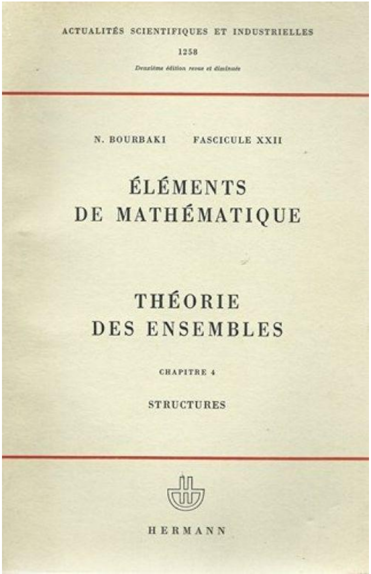

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Association des collaborateurs de Nicolas Bourbaki

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School Mathematics Project (UK)/New Mathematics (USA):

- Response to the launch of Sputnik I in 1957
- Traditional school arithmetic and geometry replaced by abstract algebra, matrices, symbolic logic, . . . — in short, mathematical topics based on set theory
- Much debate — now usually regarded as a passing fad

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- Our modern perception of real numbers took well over 2000 years to crystallise, with geometric, arithmetic, set-theoretic intuitions to the fore at different times.
- The concept of set emerged at about the same time as the modern concept of real number, 1870–1890.
- This coincidence is no coincidence.

Further reading on the development of analysis