B01 History of Mathematics
Lecture XIII
Complex analysis
Part 1: Complex numbers

MT 2020 Week 7

Summary

Part 1

- Complex numbers: validity and representation
- Substitution of complex values for real

Part 2

- Cauchy's contributions
- Riemann
- What *is* an analytic function?

Early ideas about complex numbers

Before 1600, very faint beginnings:

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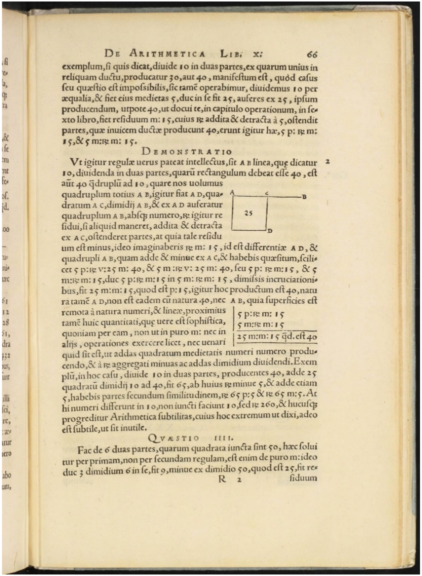
But:

> *For the most part such roots were ignored: negative roots were described merely as 'false', but complex roots as 'impossible'.* (*Mathematics emerging*, p. 459.)

Cardano and complex numbers



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DEMONSTRATIO

Vt igitur regulæ uerus pateat intellectus,ſit A B linea,quę dicatur 10, diuidenda in duas partes,quarũ rectangulum debeat eſſe 40 , eſt aũt 40 q̃druplũ ad 10 , quare nos uolumus quadruplum totius A B,igitur fiat A D,quadratum A C,dimidij A B,& ex A D auferatur quadruplum A B,abſq; numero,℞ igitur reſidui,ſi aliquid maneret, addita & detracta ex A C,oſtenderet partes,at quia tale reſiduum eſt minus,ideo imaginaberis ℞ m: 15 , id eſt differentiæ A D , & quadrupli A B,quam adde & minue ex A C,& habebis quæſitum,ſcilicet 5 p:℞ v:25 m: 40, & 5 m :℞ v: 25 m: 40, ſeu 5 p: ℞ m:15 , & 5 m:℞ m: 15,duc 5 p:℞ m:15 in 5 m: ℞ m: 15 , dimiſsis incruciationibus,fit 25 m:m: 15,quod eſt p:15,igitur hoc productum eſt 40,natura tamẽ A D,non eſt eadem cũ natura 40,nec A B, quia ſuperficies eſt remota à natura numeri,& lineæ,proximius tamẽ huic quantitati,quę uere eſt ſophiſtica, quoniam per eam , non ut in puro m: nec in alijs , operationes exercere licet , nec uenari quid ſit eſt,ut addas quadratum medietatis numeri numero producendo,& à ℞ aggregati minuas ac addas dimidium diuidendi.Exemplũ,in hoc caſu , diuide 10 in duas partes, producentes 40, adde 25 quadratũ dimidij 10 ad 40,fit 65,ab huius ℞ minue 5,& adde etiam 5,habebis partes ſecundum ſimilitudinem,℞ 65 p:5 & ℞ 65 m:5.At hi numeri differunt in 10,non iuncti faciunt 10,ſed ℞ 260,& hucuſq; progreditur Arithmetica ſubtilitas,cuius hoc extremum ut dixi,adeo eſt ſubtile,ut ſit inutile.

5 p:℞ m: 15
5 m:℞ m: 15
25 m:m: 15 q̃d. eſt 40

QVÆSTIO IIII.

Fac de 6 duas partes,quarum quadrata iuncta ſint 50, hæc ſoluitur per primam,non per ſecundam regulam,eſt enim de puro m:ideo duc 3 dimidium 6 in ſe,fit 9,minue ex dimidio 50,quod eſt 25,fit reſiduum

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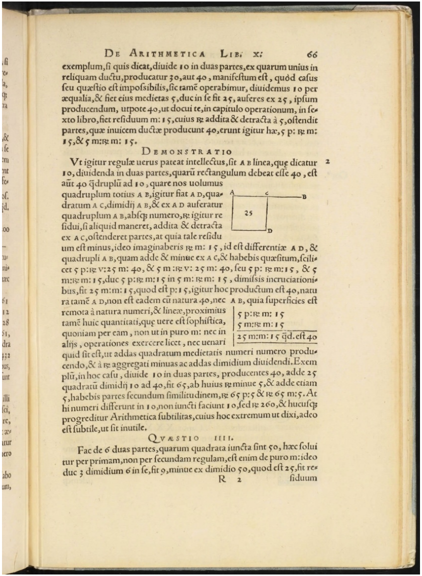
Problem: find two numbers that add to 10 and multiply to 40,

Cardano and complex numbers



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Demonstratio

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarū rectangulum debeat esse 40, est aūt 40 q̃druplū ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, R̸ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis R̸ m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: R̸ v: 25 m: 40, & 5 m: R̸ v: 25 m: 40, seu 5 p: R̸ m: 15, & 5 m: R̸ m: 15, duc 5 p: R̸ m: 15 in 5 m: R̸ m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamē A D, non est eadem cū natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamē huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à R̸ aggregati minuas ac addas dimidium diuidendi. Exemplū, in hoc casu, diuide 10 in duas partes, producentes 40, adde 25 quadratū dimidij 10 ad 40, fit 65, ab huius R̸ minue 5, & adde etiam 5, habebis partes secundum similitudinem, R̸ 65 p: 5 & R̸ 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, sed R̸ 260, & hucusq; progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

5 p: R̸ m: 15
5 m: R̸ m: 15
25 m: m: 15 q̃d. est 40

Qvæstio IIII.

Fac de 6 duas partes, quarum quadrata iuncta sint 50, hæc soluitur per primam, non per secundam regulam, est enim de puro m: ideo duc 3 dimidium 6 in se, fit 9, minue ex dimidio 50, quod est 25, fit residuum

R 2

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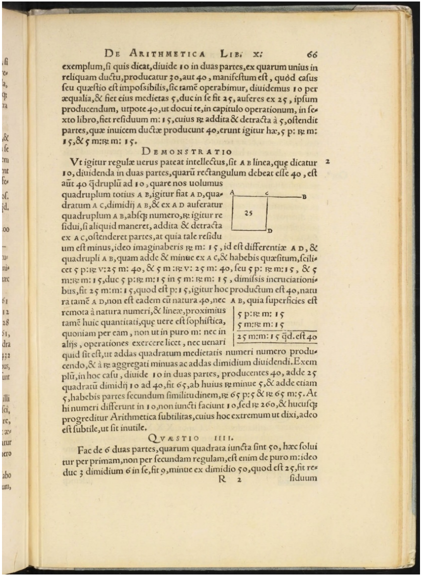
Cardano and complex numbers



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Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarũ rectangulum debeat esse 40, est aũt 40 q̃druplũ ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, ℞ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis ℞ m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, seu 5 p: ℞ m: 15, & 5 m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamẽ A D, non est eadem cũ natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamẽ huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exemplũ, in hoc casu, diuide 10 in duas partes, producentes 40, adde 25 quadratũ dimidij 10 ad 40, fit 65, ab huius ℞ minue 5, & adde etiam 5, habebis partes secundum similitudinem, ℞ 65 p: 5 & ℞ 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, sed ℞ 260, & hucusq; progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

5 p: ℞ m: 15
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Fac de 6 duas partes, quarum quadrata iuncta sint 50, hæc soluitur per primam, non per secundam regulam, est enim de puro m: ideo duc 3 dimidium 6 in se, fit 9, minue ex dimidio 50, quod est 25, fit residuum

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Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type 'square plus number equals thing'

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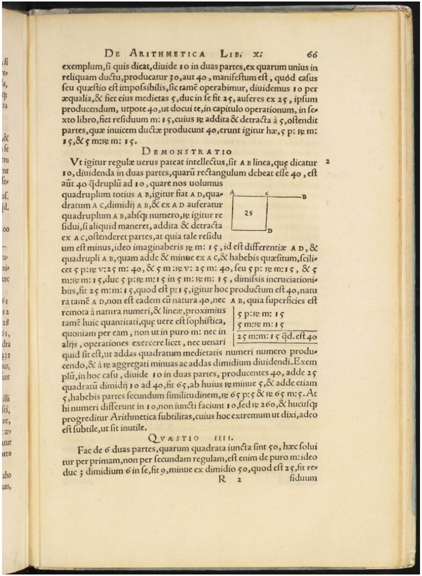

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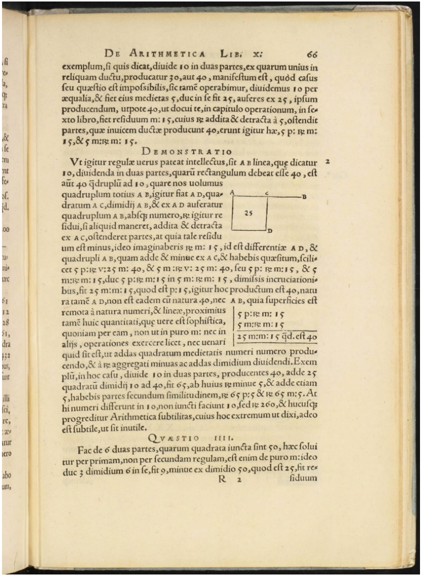

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or
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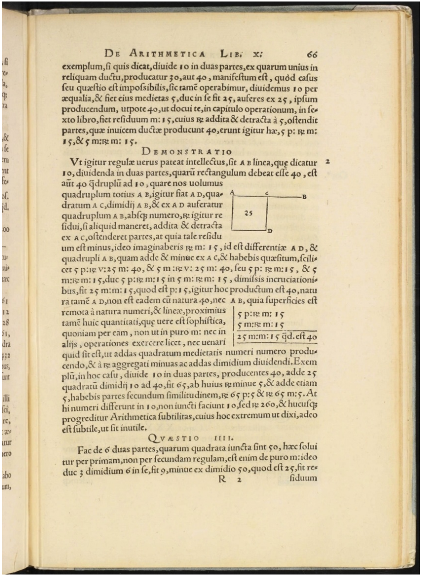
Cardano and complex numbers



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A, C, B, 25, D

5 p: ℞ m: 15
5 m: ℞ m: 15
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Fac de 6 duas partes, quarum quadrata iuncta ſint 50, hæc ſoluitur per primam, non per ſecundam regulam, eſt enim de puro m: ideo duc 3 dimidium 6 in ſe, fit 9, minue ex dimidio 50, quod eſt 25, fit reſiduum

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or
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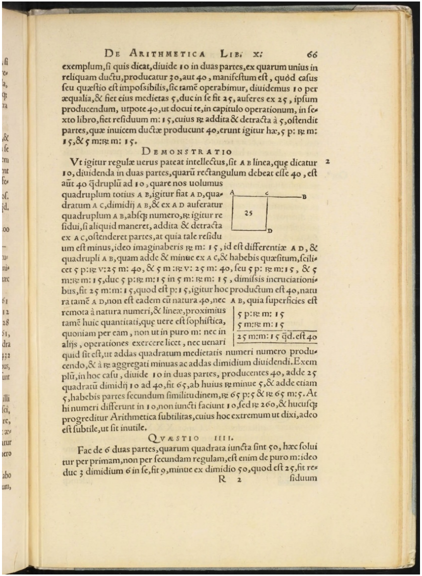
Cardano and complex numbers



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exemplum, ſi quis dicat, diuide 10 in duas partes, ex quarum unius in reliquam ductu, producatur 30, aut 40, manifeſtum eſt, quòd caſus ſeu quæſtio eſt impoſsibilis, ſic tamẽ operabimur, diuidemus 10 per æqualia, & fiet eius medietas 5, duc in ſe fit 25, auferes ex 25, ipſum producendum, utpote 40, ut docui te, in capitulo operationum, in ſexto libro, fiet reſiduum m: 15, cuius R addita & detracta à 5, oſtendit partes, quæ inuicem ductæ producunt 40, erunt igitur hæ, 5 p: R m: 15, & 5 m: R m: 15.

Demonstratio

Vt igitur regulæ uerus pateat intellectus, ſit A B linea, quę dicatur 10, diuidenda in duas partes, quarũ rectangulum debeat eſſe 40, eſt aũt 40 q̃druplũ ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, abſq; numero, R igitur reſidui, ſi aliquid maneret, addita & detracta ex A C, oſtenderet partes, at quia tale reſiduum eſt minus, ideo imaginaberis R m: 15, id eſt differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæſitum, ſcilicet 5 p: R v: 25 m: 40, & 5 m: R v: 25 m: 40, ſeu 5 p: R m: 15, & 5 m: R m: 15, duc 5 p: R m: 15 in 5 m: R m: 15, dimiſsis incruciationibus, fit 25 m: m: 15, quod eſt p: 15, igitur hoc productum eſt 40, natura tamẽ A D, non eſt eadem cũ natura 40, nec A B, quia ſuperficies eſt remota à natura numeri, & lineæ, proximius tamẽ huic quantitati, quę uere eſt ſophiſtica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid ſit eſt, ut addas quadratum medietatis numeri numero producendo, & à R aggregati minuas ac addas dimidium diuidendi. Exemplũ, in hoc caſu, diuide 10 in duas partes, producentes 40, adde 25 quadratũ dimidij 10 ad 40, fit 65, ab huius R minue 5, & adde etiam 5, habebis partes ſecundum ſimilitudinem, R 65 p: 5 & R 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, ſed R 260, & hucuſq; progreditur Arithmetica ſubtilitas, cuius hoc extremum ut dixi, adeo eſt ſubtile, ut ſit inutile.

5 p: R m: 15
5 m: R m: 15
25 m: m: 15 q̃d. eſt 40

Quæstio IIII.

Fac de 6 duas partes, quarum quadrata iuncta ſint 50, hæc ſoluitur per primam, non per ſecundam regulam, eſt enim de puro m: ideo duc 3 dimidium 6 in ſe, fit 9, minue ex dimidio 50, quod eſt 25, fit reſiduum

R 2

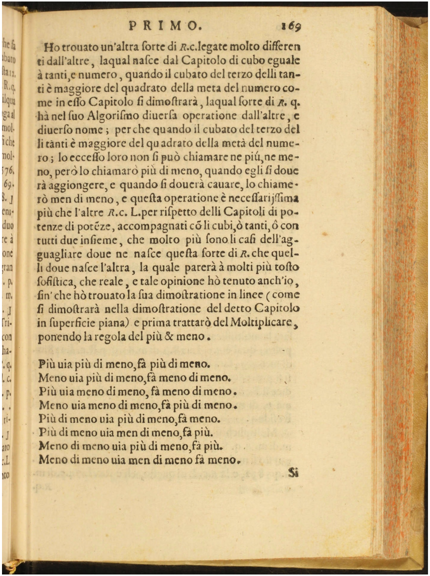
Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type 'square plus number equals thing'

Cardano noted that $5+\sqrt{-15}$ and $5-\sqrt{-15}$ solve the problem, "dismissis incruciationibus", meaning

"putting aside mental tortures",
or
"the cross-multiples having canceled out",
or
"the imaginary part being lost"

But regarded such ideas as absurd and useless

Bombelli and complex numbers



PRIMO. 169

Ho trouato un'altra ſorte di R.c.legate molto differenti dall'altre, laqual naſce dal Capitolo di cubo eguale à tanti, e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R. q. hà nel ſuo Algoriſmo diuerſa operatione dall'altre, e diuerſo nome; per che quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo ecceſſo loro non ſi può chiamare ne più, ne meno, però lo chiamarò più di meno, quando egli ſi dourà aggiongere, e quando ſi dourà cauare, lo chiamerò men di meno, e queſta operatione è neceſſarijſſima più che l'altre R.c. L. per riſpetto delli Capitoli di potenze di potēze, accompagnati cō li cubi, ò tanti, ò con tutti due inſieme, che molto più ſono li caſi dell'agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l'altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch'io, ſin' che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
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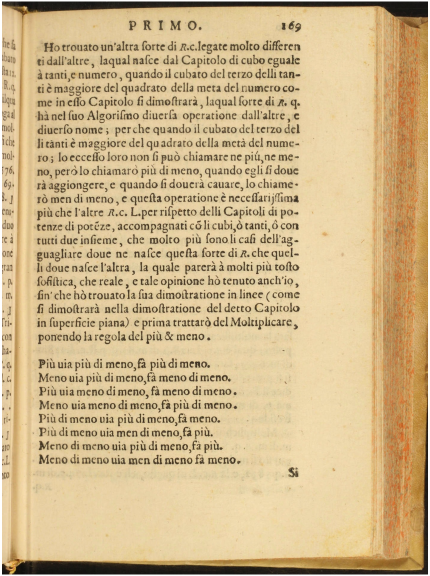
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Si

"Another sort of cube root much different from the former . . ."

Bombelli and complex numbers



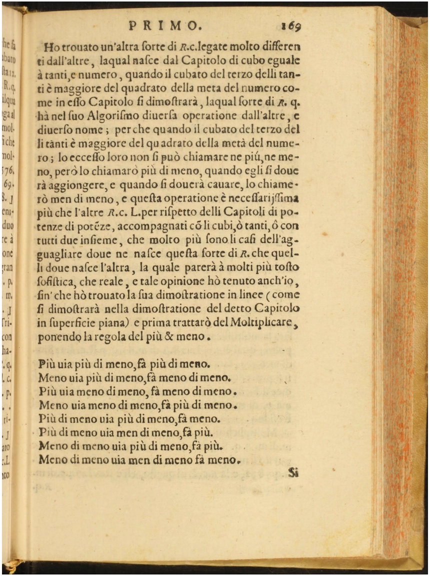
"Another sort of cube root much different from the former ..."

Systematic rules:

più di meno via più di meno, fà meno $(\sqrt{-1} \times \sqrt{-1} = -1)$

meno di meno via più di meno, fà più $(-\sqrt{-1} \times \sqrt{-1} = 1)$

Bombelli and complex numbers



PRIMO. 169

Ho trouato un'altra sorte di R.c.legate molto differenti dall'altre, laqual nasce dal Capitolo di cubo eguale à tanti, e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in esso Capitolo si dimostrarà, laqual sorte di R. q. hà nel suo Algorismo diuersa operatione dall'altre, e diuerso nome; per che quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo eccesso loro non si può chiamare ne più, ne meno, però lo chiamarò più di meno, quando egli si douerà aggiongere, e quando si douerà cauare, lo chiamerò men di meno, e questa operatione è necessarijssima più che l'altre R.c. L. per rispetto delli Capitoli di potẽze di potẽze, accompagnati cõ li cubi, ò tanti, ò con tutti due insieme, che molto più sono li casi dell'agguagliare doue ne nasce questa sorte di R. che quelli doue nasce l'altra, la quale parerà à molti più tosto sofistica, che reale, e tale opinione hò tenuto anch'io, sin' che hò trouato la sua dimostratione in linee (come si dimostrarà nella dimostratione del detto Capitolo in superficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

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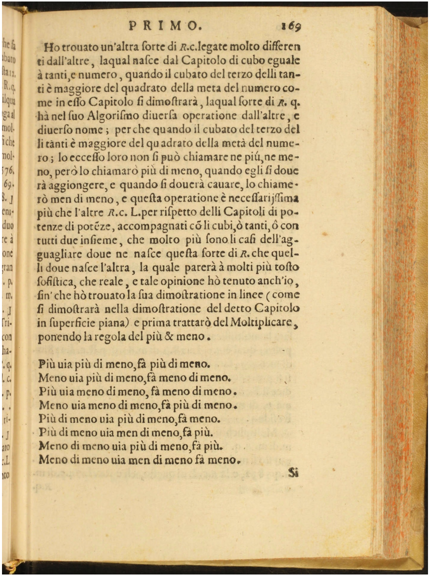
But complex numbers were not admitted as solutions of equations — they could appear in calculations, provided they cancelled out by the end

Bombelli and complex numbers



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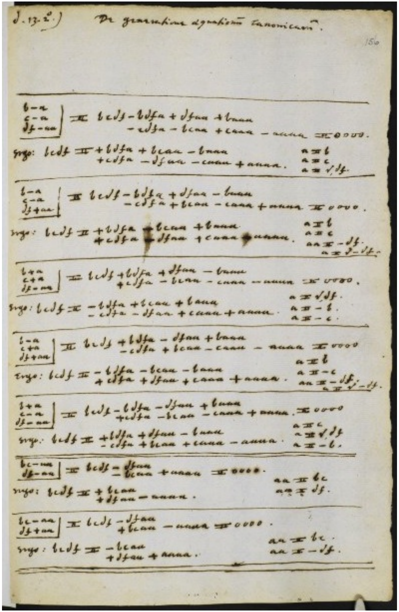
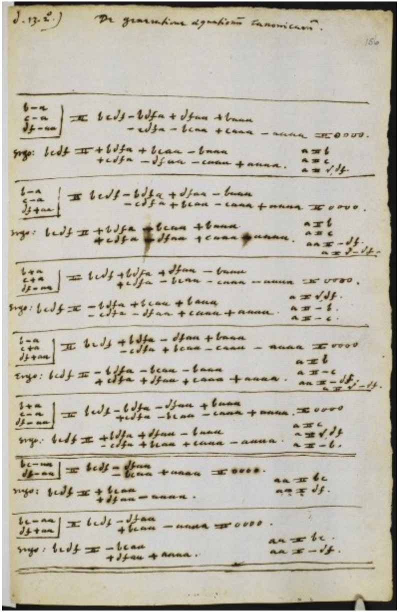
Complex numbers justified through practical use?

Harriot and complex numbers



Add MS 6783 f. 156

Harriot and complex numbers



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Unpublished manuscripts contain systematic treatment of complex roots of equations

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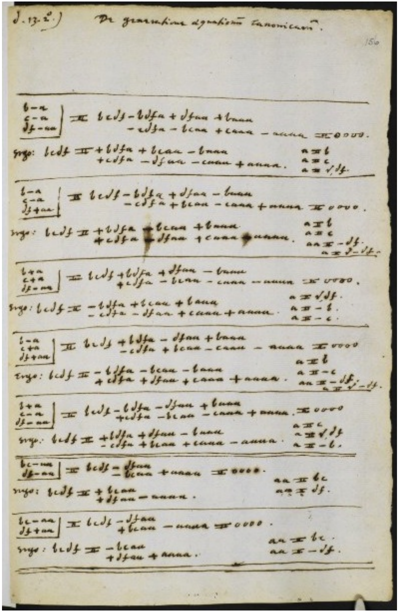

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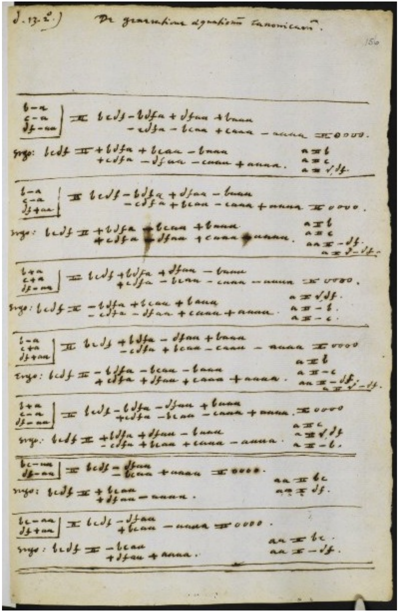

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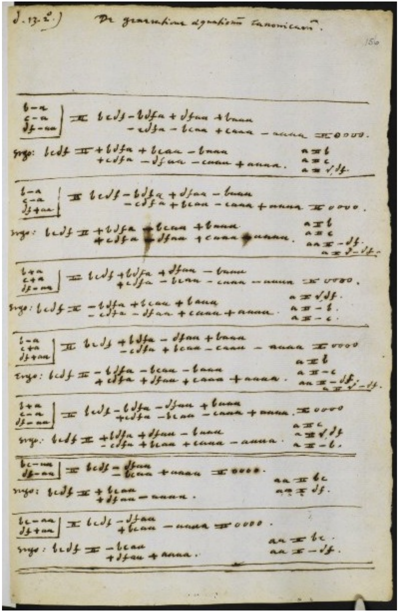

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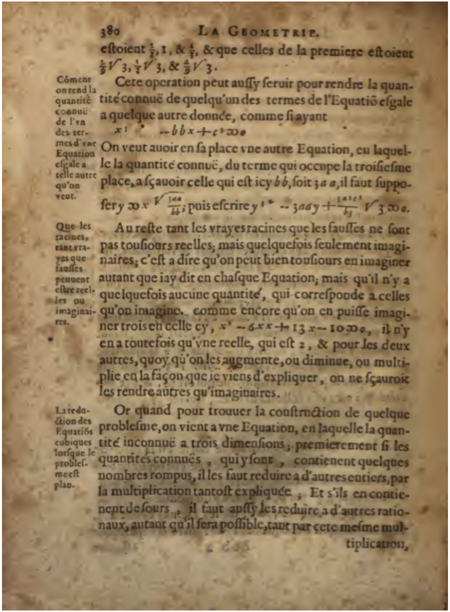
Cf. Harriot's *Artis analyticae praxis* (1931), pp. 14–15; see:

Muriel Seltman & Robert Goulding, *Thomas Harriot's Artis analyticae praxis: an English translation with commentary*, Springer, 2007

Descartes and 'imaginaries'



380 LA GEOMETRIE.

estoient $\frac{2}{3}$, 1, & $\frac{4}{3}$, & que celles de la premiere estoient $\frac{2}{9}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$, & $\frac{4}{9}\sqrt{3}$.

Cõment on rend la quantité connuë de l'vn des termes d'vne Equation esgale a telle autre qu'on veut.

Cete operation peut aussy seruir pour rendre la quantité connuë de quelqu'un des termes de l'Equatiõ esgale a quelque autre donnée, comme si ayant

$x^3 * - bbx + c^3 \infty 0$

On veut auoir en sa place vne autre Equation, en laquelle la quantité connuë, du terme qui occupe la troisiesme place, a sçauoir celle qui est icy bb, soit $3aa$, il faut supposer $y \infty x\sqrt{\frac{3aa}{bb}}$; puis escrire $y^3 * - 3aay + \frac{3a^3c^3}{b^3}\sqrt{3} \infty 0$.

Que les racines, tant vrayes que fausses peuuent estre reelles ou imaginaires.

Au reste tant les vrayes racines que les fausses ne sont pas tousiours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien tousiours en imaginer autant que iay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine. comme encore qu'on en puisse imaginer trois en celle cy, $x^3 - 6xx + 13x - 10 \infty 0$, il n'y en a toutefois qu'vne reelle, qui est 2, & pour les deux autres, quoy qu'on les augmente, ou diminue, ou multiplie en la façon que ie viens d'expliquer, on ne sçauroit les rendre autres qu'imaginaires.

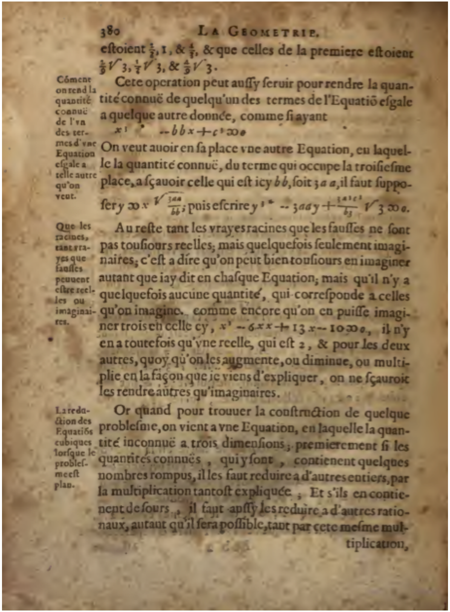
La reduction des Equatiõs cubiques lorsque le problesme est plan.

Or quand pour trouuer la construction de quelque problesme, on vient a vne Equation, en laquelle la quantité inconnuë a trois dimensions; premierement si les quantités connuës, qui y sont, contienent quelques nombres rompus, il les faut reduire a d'autres entiers, par la multiplication tantost expliquée; Et s'ils en contienent de sours, il faut aussy les reduire a d'autres rationaux, autant qu'il sera possible, tant par cete mesme multiplication,

La géométrie (1637):

introduced the term 'imaginaire'

Descartes and ‘imaginaries’



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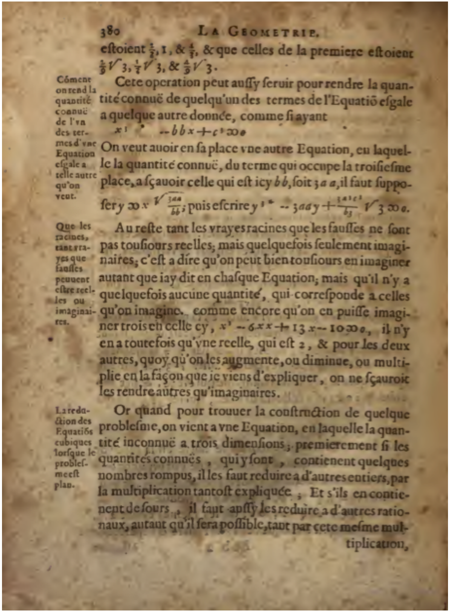

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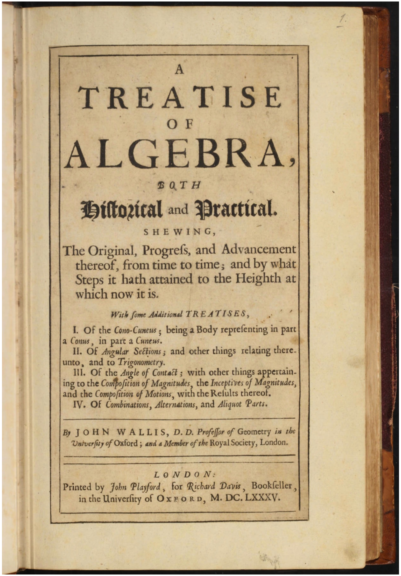
Didn't regard them as numbers

Ideas about complex numbers in the later 17th century

John Wallis, *A treatise of algebra* (1685): complex numbers based on insights derived from

- Euclidean geometry
- trigonometry
- properties of conics

(See: *Mathematics emerging*, §15.1.1.)



A

TREATISE

OF

ALGEBRA,

BOTH

Hiſtorical and Practical.

SHEWING,

The Original, Progreſs, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With ſome Additional TREATISES,

I. Of the *Cono-Cuneus*; being a Body repreſenting in part a *Conus*, in part a *Cuneus*.

II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.

III. Of the *Angle of Contact*; with other things appertaining to the *Compoſition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Compoſition of Motions*, with the Reſults thereof.

IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

By JOHN WALLIS, *D.D. Profeſſor of* Geometry *in the Univerſity of* Oxford; *and a Member of the* Royal Society, London.

LONDON:

Printed by *John Playford*, for *Richard Davis*, Bookſeller, in the Univerſity of OXFORD, M. DC. LXXXV.

Wallis: justification of imaginary numbers

- A man starts at A and walks 5 yds to B, then retreats 2 yds to C: overall, he has covered 3 yds.

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- Somewhere on the seashore, we gain 26 units of land from the sea, but lose 10 units.

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- Somewhere on the seashore, we gain 26 units of land from the sea, but lose 10 units. Thus, we have gained 16 units overall; if this is a perfect square, then it has side 4 units of length.
- If instead we lose 26 units of land, but gain 10, then we have lost 16 units overall, or gained -16.

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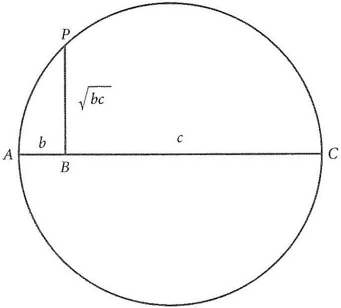
- A man starts at A and walks 5 yds to B, then retreats 2 yds to C: overall, he has covered 3 yds. If he instead retreats 8 yds to D, then we may say that he has covered -3 yds.
- Somewhere on the seashore, we gain 26 units of land from the sea, but lose 10 units. Thus, we have gained 16 units overall; if this is a perfect square, then it has side 4 units of length.
- If instead we lose 26 units of land, but gain 10, then we have lost 16 units overall, or gained -16. The area in question (assumed to be a square) might therefore be viewed as having side $\sqrt{-16}$.

(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 184–185)

Wallis: imaginary numbers as geometric means



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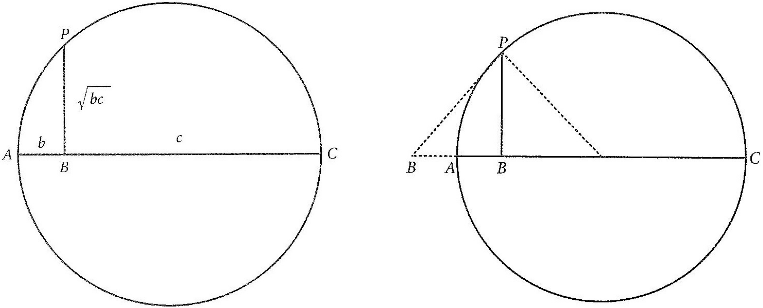
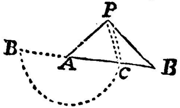


(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 185–186)

"A new Impossibility in Algebra"

John Wallis, *A treatise of algebra*, p. 267 'Of negative squares':
... requires a new Impossibility in Algebra

Suppoſe again, $AP = 15$, $PC = 12$, (and therefore $AC = \sqrt{:225 - 144:} = \sqrt{81} = 9$,) $PB = 20$ (and therefore $BC = \sqrt{:400 - 144:} = \sqrt{256} = +16$, or -16:) Then is $AB = 9 + 16 = 25$, or $AB = 9 - 16 = -7$. The one Affirmative, the other Negative. (The ſame values would be, but with contrary Signs, if we take $AC = \sqrt{81} = -9$: That is, $AB = -9 + 16 = +7$, $AB = -9 - 16 = -25$.)



⋮

Which gives indeed (as before) a double value of AB, $\sqrt{175}, + \sqrt{-81}$, and $\sqrt{175}, - \sqrt{-81}$: But ſuch as requires a new Impoſſibility in Algebra, (which in Lateral Equations doth not happen;) not that of a Negative Root, or a Quantity leſs than nothing; (as before,) but the Root of a Negative Square. Which in ſtrictneſs of ſpeech, cannot be: ſince that no Real Root (Affirmative or Negative,) being Multiplied into itſelf, will make a Negative Square.

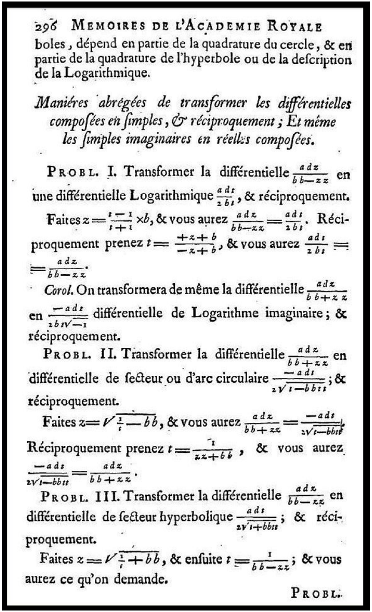
Complex numbers in the 18th century (1)

Nature remained unclear:

“that amphibian between being and not-being, which we call the imaginary root of negative unity” (Leibniz, 1702)

But complex numbers were increasingly being used . . .

Complex numbers in the 18th century (2)



boles, dépend en partie de la quadrature du cercle, & en partie de la quadrature de l'hyperbole ou de la description de la Logarithmique.

Maniéres abrégées de transformer les différentielles composées en simples, & réciproquement; Et même les simples imaginaires en réelles composées.

PROBL. I. Transformer la différentielle $\frac{adz}{bb-zz}$ en une différentielle Logarithmique $\frac{adt}{2bt}$, & réciproquement.

Faites $z = \frac{t-1}{t+1} \times b$, & vous aurez $\frac{adz}{bb-zz} = \frac{adt}{2bt}$. Réciproquement prenez $t = \frac{+z+b}{-z+b}$, & vous aurez $\frac{adt}{2bt} = \frac{adz}{bb-zz}$.

Corol. On transformera de même la différentielle $\frac{adz}{bb+zz}$ en $\frac{-adt}{2bt\sqrt{-1}}$ différentielle de Logarithme imaginaire; & réciproquement.

PROBL. II. Transformer la différentielle $\frac{adz}{bb+zz}$ en différentielle de secteur ou d'arc circulaire $\frac{-adt}{2\sqrt{t-bbtt}}$; & réciproquement.

Faites $z = \sqrt{\frac{1}{t} - bb}$, & vous aurez $\frac{adz}{bb+zz} = \frac{-adt}{2\sqrt{t-bbtt}}$. Réciproquement prenez $t = \frac{1}{zz+bb}$, & vous aurez $\frac{-adt}{2\sqrt{t-bbtt}} = \frac{adz}{bb+zz}$.

PROBL. III. Transformer la différentielle $\frac{adz}{bb-zz}$ en différentielle de secteur hyperbolique $\frac{adt}{2\sqrt{t+bbtt}}$; & réciproquement.

Faites $z = \sqrt{\frac{1}{t} + bb}$, & ensuite $t = \frac{1}{bb-zz}$; & vous aurez ce qu'on demande.

PROBL.

Johann Bernoulli, 'Solution d'un problème concernant le calcul intégrale, ...', *Mémoires de l'Académie royale des sciences*, 1702:

Complex numbers in the 18th century (2)

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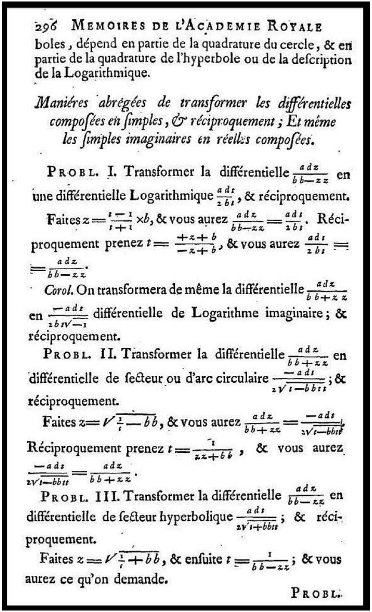
Complex numbers in the 18th century (2)



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No worries about the validity of switching between real and complex integrals

(See *Mathematics emerging*, §15.2.1)

Complex numbers in the 18th century (3)

How ÆQUATIONS *are to be ſolv'd.*

AFTER therefore in the Solution of a Queſtion you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are ſuppos'd given, are really given in Numbers, thoſe Numbers are to be ſubſtituted in their room in the Æquation, and you'll have a Numeral Æquation, whoſe Root being extracted will ſatisfy the Queſtion. As if in the Diviſion of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Caſe the Radius r is given in Numbers, and the Line q ſubtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I ſubſtitute thoſe Numbers in the Æquation for r and q, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line ſubtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an Æquation.

But the Root is a Number which being ſubſtituted in the Æquation for the Letter or Species ſignifying the Root, will make all the Terms vaniſh. Thus Unity is the Root of the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, becauſe being writ for x it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. And thus, if for x you write the Number 3, or the Negative Number -5, and in both Caſes there will be produc'd nothing, the Affirmative and Negative Terms in theſe four Caſes deſtroying one another; then ſince any of the Numbers written in the Æquation fulfils the Condition of x, by making all the Terms of the Æquation together equal to nothing, any of them will be the Root of the Æquation.

And that you may not wonder that the ſame Æquation may have ſeveral Roots, you muſt know that there may be more Solutions [than one] of the ſame Problem. As if there was ſought the Interſection of two given Circles; there are two Interſections, and conſequently the Queſtion admits two Anſwers; and then the Æquation determining the

Isaac Newton, *Universal Arithmetick*, 1728:

p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible"

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p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible" — complex numbers as an indicator of real-world solvability of problems

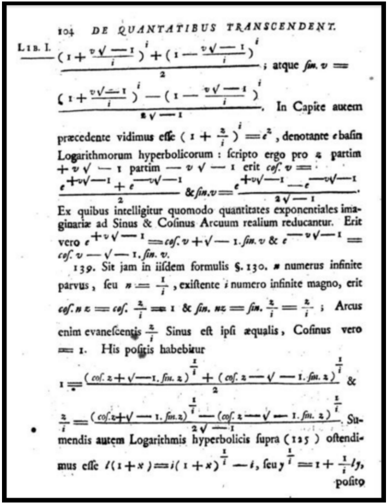
Complex numbers in the 18th century (4)

Leonhard Euler also used them freely: e.g., in *Introductio in analysin infinitorum*, 1748, §138:

$$e^{+v\sqrt{-1}} = \cos.v + \sqrt{-1}.\sin.v$$

$$e^{-v\sqrt{-1}} = \cos.v - \sqrt{-1}.\sin.v$$

(See *Mathematics emerging*, §9.2.3)



104 *DE QUANTATIBUS TRANSCENDENT.*

LIB. I. $\frac{(1+\frac{v\sqrt{-1}}{i})^i + (1-\frac{v\sqrt{-1}}{i})^i}{2}$; atque *ſin.* $v =$ $\frac{(1+\frac{v\sqrt{-1}}{i})^i - (1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}$. In Capite autem præcedente vidimus eſſe $(1+\frac{z}{i})^i = e^z$, denotante e baſin Logarithmorum hyperbolicorum: ſcripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit *coſ.* $v =$ $\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$ & *ſin.* $v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$. Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Coſinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = cos.v + \sqrt{-1}.sin.v$ & $e^{-v\sqrt{-1}} = cos.v - \sqrt{-1}.sin.v$.

139. Sit jam in iiſdem formulis §. 130. n numerus infinite parvus, ſeu $n = \frac{1}{i}$, exiſtente i numero infinite magno, erit *coſ.* $nz =$ *coſ.* $\frac{z}{i} = 1$ & *ſin.* $nz =$ *ſin.* $\frac{z}{i} = \frac{z}{i}$; Arcus enim evaneſcentis $\frac{z}{i}$ Sinus eſt ipſi æqualis, Coſinus vero $= 1$. His poſitis habebitur

$$1 = \frac{(cos.z + \sqrt{-1}.sin.z)^{\frac{1}{i}} + (cos.z - \sqrt{-1}.sin.z)^{\frac{1}{i}}}{2} \ \&$$

$$\frac{z}{i} = \frac{(cos.z + \sqrt{-1}.sin.z)^{\frac{1}{i}} - (cos.z - \sqrt{-1}.sin.z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$ Sumendis autem Logarithmis hyperbolicis ſupra (125) oſtendimus eſſe $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, ſeu $y^{\frac{1}{i}} = 1 + \frac{1}{i}ly$,

poſito

The Fundamental Theorem of Algebra

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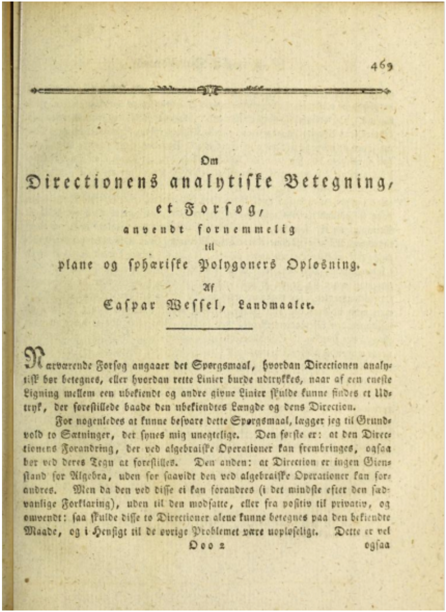
- 1821: Argand's proof appears in Cauchy's *Cours d'analyse*

New ways of viewing complex numbers



469

Om

Directionens analytiske Betegning,

et Forsøg,

anvendt fornemmelig

til

plane og sphæriske Polygoners Opløsning.

Af

Caspar Wessel, Landmaaler.

Nærværende Forsøg angaaer det Spørgsmaal, hvordan Directionen analytisk bør betegnes, eller hvordan rette Linier burde udtrykkes, naar af een eneste Ligning mellem een ubekiendt og andre givne Linier skulde kunne findes et Udtryk, der forestillede baade den ubekiendtes Længde og dens Direction.

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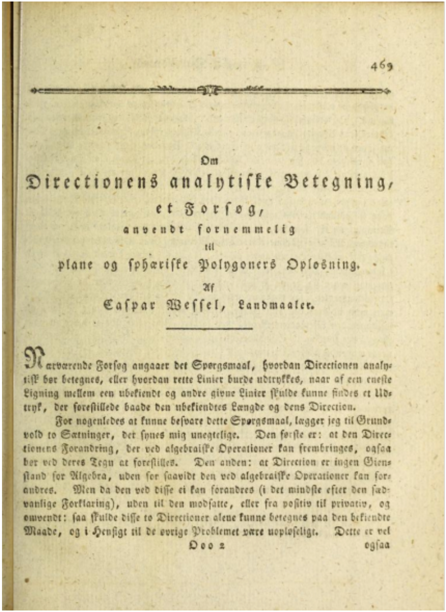
Ooo 2 ogsaa

Caspar Wessel, ‘Om Directionens analytiske Betegning . . .’ [‘On the analytic representation of direction . . .’], *Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter*, 1799

New ways of viewing complex numbers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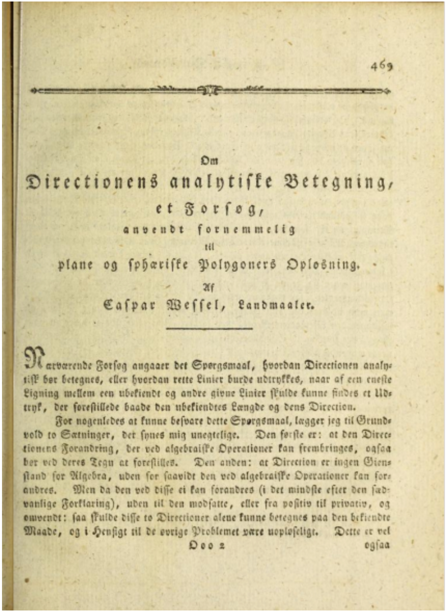

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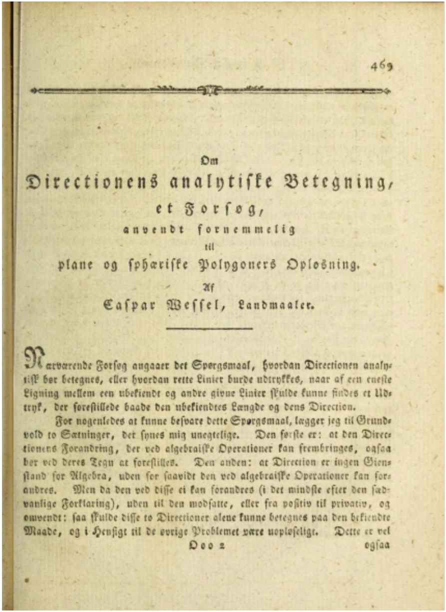
Published in Danish — not well known

New ways of viewing complex numbers


469

Om
Directionens analytiske Betegning,
et Forsøg,
anvendt fornemmelig
til
plane og sphæriske Polygoners Opløsning.
Af
Caspar Wessel, Landmaaler.

Nærværende Forsøg angaaer det Spørgsmaal, hvordan Directionen analytisk bør betegnes, eller hvordan rette Linier burde udtrykkes, naar af een eneste Ligning mellem een ubekiendt og andre givne Linier skulde kunne findes et Udtryk, der forestillede baade den ubekiendtes Længde og dens Direction.

For nogenledes at kunne besvare dette Spørgsmaal, lægger jeg til Grundvold to Sætninger, det synes mig unegtelige. Den første er: at den Directionens Forandring, der ved algebraiske Operationer kan frembringes, ogsaa bør ved deres Tegn at forestilles. Den anden: at Direction er ingen Gienstand for Algebra, uden for saavidt den ved algebraiske Operationer kan forandres. Men da den ved disse ei kan forandres (i det mindste efter den sædvanlige Forklaring), uden til den modsatte, eller fra positiv til privativ, og omvendt: saa skulde disse to Directioner alene kunne betegnes paa den bekiendte Maade, og i Hensigt til de øvrige Problemet være uopløselige. Dette er vel

Ooo 2 ogsaa

Caspar Wessel, 'Om Directionens analytiske Betegning ...' ['On the analytic representation of direction ...'], *Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter*, 1799

Published in Danish — not well known

French translation published in 1897

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ESSAI

SUR UNE MANIÈRE DE REPRÉSENTER

LES QUANTITÉS IMAGINAIRES

DANS

LES CONSTRUCTIONS GÉOMÉTRIQUES,

PAR R. ARGAND.

2e ÉDITION

PRÉCÉDÉE D'UNE PRÉFACE

PAR M. J. HOÜEL

ET SUIVIE D'UN APPENDICE

Contenant des Extraits des *Annales de Gergonne*, relatifs à la question des imaginaires.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

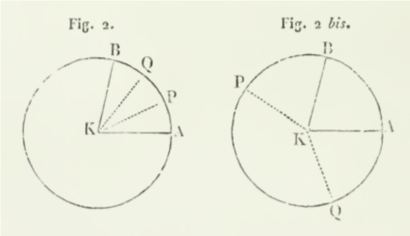
SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1874

(Tous droits réservés.)

Robert Argand, *Essay on a method of representing imaginary quantities* . . ., 1806



New ways of viewing complex numbers

Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.

By WILLIAM ROWAN HAMILTON,

M. R. I. A., F. R. A. S., Hon. M. R. S. Ed. and Dub., Fellow of the American Academy of Arts and Sciences, and of the Royal Northern Antiquarian Society at Copenhagen, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.

Read November 4th, 1833, and June 1st, 1835.

General Introductory Remarks.

The Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari*, or the *sapere,*) is eminently prized and sought for. The Practical person seeks a Rule which he may apply, the Philological person seeks a Formula which he may write, the Theoretical person seeks a Theorem on which he may meditate. The felt imperfections of Algebra are of three answering kinds. The Practical Algebraist complains of imperfection when he finds his Instrument limited in power; when a rule, which he could happily apply to many cases, can be hardly or not at all applied by him to some new case; when it fails to enable him to do or to discover something else, in some other Art, or in some other Science, to which Algebra with him was but subordinate, and for the sake of which and not for its own sake, he studied Algebra. The Philological Algebraist complains of imperfection, when his Language presents him with an Anomaly; when he finds an Exception disturb the simplicity of his Notation, or the symmetrical structure of his Syntax; when a Formula must be written with precaution, and a Symbolism is not universal. The Theoretical Algebraist complains of imperfection, when the clearness of his Contemplation is obscured; when the Reasonings of his Science seem anywhere to oppose each other, or become in any part too complex or too little valid for his belief to rest firmly upon them; or when, though trial may have taught him that a rule is useful, or that a formula gives true results, he cannot prove that rule, nor understand that formula: when he cannot rise to intuition from induction, or cannot look beyond the signs to the things signified.

Transactions of the Royal Irish Academy, 1837

Complex numbers as ordered pairs subject to specified rules:

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$$(a, b) \pm (c, d) = (a \pm c, b \pm d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc)$$
$$\frac{(a, b)}{(c, d)} = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

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$$(a,b)\pm(c,d)=(a\pm c, b\pm d)$$

$$(a,b)(c,d)=(ac-bd, ad+bc)$$

$$\frac{(a,b)}{(c,d)}=\left(\frac{ac+bd}{c^2+d^2},\frac{bc-ad}{c^2+d^2}\right)$$

Led to the search for triples, and thence to quaternions