BO1 History of Mathematics
Lecture XIV
Linear algebra
Part 1: Linear equations

MT 2020 Week 7

Summary

Part 1

- Linear equations

Part 2

- Determinants
- Eigenvalues
- Matrices

Part 3

- Vector spaces

Difficulties in the historical study of linear algebra

> *Linear algebra may be mathematically simple but its history is more complicated than any other topic in this book. . . . [Its development is] a very tangled tale.*

(*Mathematics Emerging*, p. 548.)

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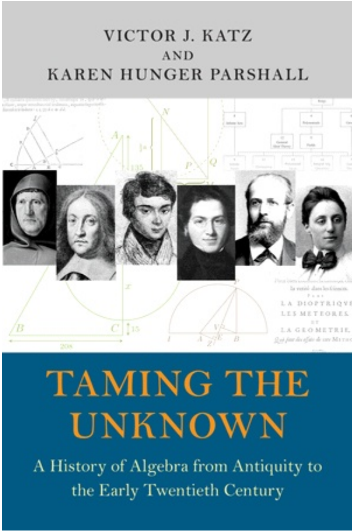
Warning: matrices (etc.) are primary in modern teaching, determinants secondary. For about 200 years until 1940 (or thereabouts) the reverse was the case: determinants came first.

On the history of linear algebra



(Princeton University Press, 2014)

Jiǔzhāng Suànshù (China, c. 150 BC)

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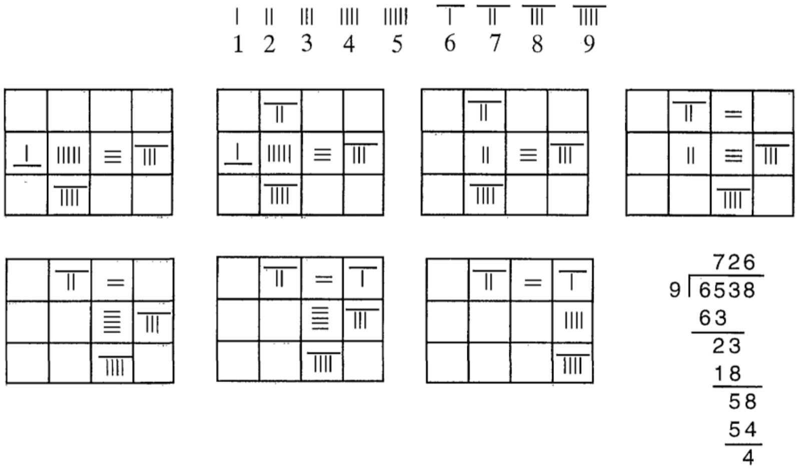
Content: calculation of areas ($\pi \approx 3.14159$), rates of exchange, computation with fractions, proportion, extraction of square and cube roots, calculation of volumes, systems of linear equations, Pythagoras' Theorem, ...

Chinese calculation



Base 10 system of rods on counting board: red for positive, black for negative

Early linear equations in China

Chapter 7: solution of pairs of equations in two unknowns by the method of false position

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There are three types of grain

3 bundles of the first, 2 of the second, and 1 of the third contain 39 measures
2 of the first, 3 of the second, and 1 of the third contain 34
1 of the first, 2 of the second, and 3 of the third contain 26

How many measures in a bundle of each type?

Early linear equations in China

Chapter 7: solution of pairs of equations in two unknowns by the method of false position

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How many measures in a bundle of each type?

Solved on a counting board by Gaussian elimination, known here as 'fāngchéng' 方程

Early linear equations in China

There are five families which share a well. 2 of A's ropes are short of the well's depth by 1 of B's ropes. 3 of B's ropes are short of the depth by 1 of C's ropes. 4 of C's ropes are short by 1 of D's ropes. 5 of D's ropes are short by 1 of E's ropes. 6 of E's ropes are short by 1 of A's ropes. Find the depth of the well and the length of each rope.

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Five equations in six unknowns, so indeterminate

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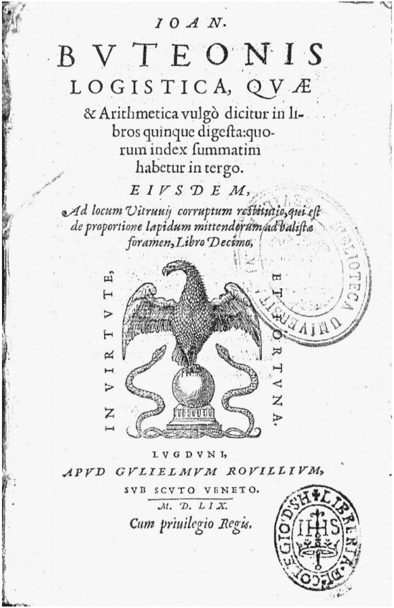
Five equations in six unknowns, so indeterminate

Liu Hui: we can only give a solution in terms of proportions of the lengths

Early linear equations in Europe


IOAN.

BVTEONIS

LOGISTICA, QVÆ

& Arithmetica vulgò dicitur in libros quinque digesta: quorum index summatim habetur in tergo.

EIVSDEM,

Ad locum Vitruuij corruptum restitutio, qui est de proportione lapidum mittendorum ad balistæ foramen, Libro Decimo.

IN VIRTVTE, ET FORTVNA.

LVGDVNI,

APVD GVLIELMVM ROVILLIVM,

SVB SCVTO VENETO.

M. D. LIX.

Cum priuilegio Regis.

Jean Borrel [Ioannes Buteus] *Logistica, quæ et Arithmetica vulgo dicitur in libros quinque digesta* (*Logistic, also known as Arithmetic, digested in five books*), 1559

Linear equations in Borrel's *Logistica*

2 *A*, 1 *B* [60 *ſingulatim in* 3, *fit* 6 *A*, 3 *B*, [180. *Ex his detrahe* 1 *A*, 3 *B* [60, *reſtat* 5 *A* [120]. *Partire in* 5, *prouenit* 24, *qui primus eſt numerus ex quæſitis. Ex numero* 30 *aufer* 24, *reſiduum fit* 6, *quod eſt dimidium ſecundi, quare ipſe eſt* 12. *Sunt igitur duo numeri* 24, & 12, *quos oportuit inuenire.*

Tres numeros inuenire, quorum primus cum triente reliquorum faciat 14. Secundus cum aliorum quadrante 8. Tertius item cum parte quinta reliquorum 8.

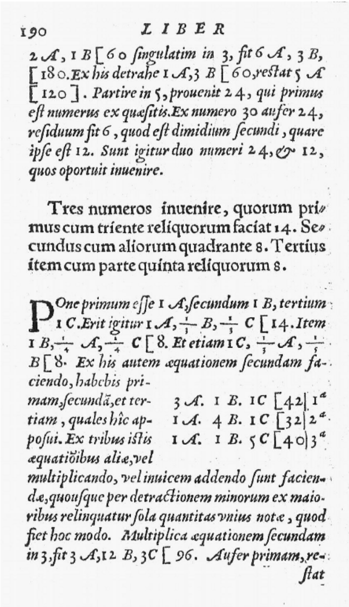
Pone primum eſſe 1 *A*, *ſecundum* 1 *B*, *tertium* 1 *C*. *Erit igitur* 1 *A*, $\frac{1}{3}$ *B*, $\frac{1}{3}$ *C* [14. *Item* 1 *B*, $\frac{1}{4}$ *A*, $\frac{1}{4}$ *C* [8. *Et etiam* 1 *C*, $\frac{1}{5}$ *A*, $\frac{1}{5}$ *B* [8. *Ex his autem æquationem ſecundam faciendo, habebis primam, ſecundã, et tertiam, quales hîc appoſui. Ex tribus iſtis æquatiõibus aliæ, vel*

3 *A*.	1 *B*.	1 *C*	[42	1ª
1 *A*.	4 *B*.	1 *C*	[32	2ª
1 *A*.	1 *B*.	5 *C*	[40	3ª

multiplicando, vel inuicem addendo ſunt faciendæ, quouſque per detractionem minorum ex maioribus relinquatur ſola quantitas vnius notæ, quod fiet hoc modo. Multiplica æquationem ſecundam in 3, *fit* 3 *A*, 12 *B*, 3 *C* [96. *Aufer primam, reſtat*

To find three numbers, of which the first with a third of the rest makes 14. The second with a quarter of the rest makes 8. Likewise the third with a fifth part of the rest makes 8.

Linear equations in Borrel's *Logistica*



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3 A. 1 B. 1 C [42	1ª
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Put the first to be $1A$, *the second* $1B$, *the third* $1C$. ...

[Derives a system of equations with '.' for addition and '[' for equality.]

Linear equations in Borrel's *Logistica*

190 LIBER

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Put the first to be 1A, *the second* 1B, *the third* 1C. ...

[Derives a system of equations with '.' for addition and '[' for equality.]

Multiply by 3, by 4 and by 5 respectively, etc.

(See *Mathematics emerging*, §17.1.1.)

More unknowns

GVL. GOS. DE ARTE

bunt 60 equalia 1 A, quare primus eſt 60, iam vero 2 B 1 C equalia fuerunt 100, tollamus 1 C hoc eſt 20, reſtabunt 80 æqualia 2 B, & 1 B eſt 40, ſuntque tres numeri quæſiti 60 40 20, quibus veſtigatis opus fuit.

Problema V.

Inueniamus quatuor numeros quorum primus cum ſemiſſe reliquorum faciat 17, ſecundus cum aliorum triente 12, tertius cum aliorum quadrante 13, quartus item cum aliorum ſextante 13.

Sint illi quatuor A B C D, & ſint 1 A $\frac{1}{2}$ B $\frac{1}{2}$ C $\frac{1}{2}$ D equalia 17, 1 B $\frac{1}{3}$ A $\frac{1}{3}$ C $\frac{1}{3}$ D equalia 12, 1 C $\frac{1}{4}$ A $\frac{1}{4}$ B $\frac{1}{4}$ D æqualia 13, 1 D $\frac{1}{6}$ A $\frac{1}{6}$ B $\frac{1}{6}$ C equalia 13, reuocentur hęc ad integros numeros, exiſtent 2 A 1 B 1 C 1 D æqualia 34, 1 A 3 B 1 C 1 D æqualia 36, 1 A 1 B 4 C 1 D æqualia 52, 1 A 1 B 1 C 6 D æqualia 78,

Guillaume Gosselin, *De arte magna seu de occulta parte numerorum quae et Algebra et Almucabala vulgo dicitur* (*On the great art or the hidden part of numbers commonly called Algebra and Almucabala*), 1577

$$1A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = 17$$
$$1B + \frac{1}{3}A + \frac{1}{3}C + \frac{1}{3}D = 12$$
$$1C + \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}D = 13$$
$$1D + \frac{1}{6}A + \frac{1}{6}B + \frac{1}{6}C = 13$$

A 17th-century example

After reading Gosselin . . .
John Pell to Sir Charles Cavendish (1646):

Exemplum . . . satis determinatis

$$
\begin{aligned}
3a - 4b + 5c &= 2 \\
5a + 3b - 2c &= 58 \\
7a - 5b + 4c &= 14
\end{aligned}
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(Solved via Pell's 'three-column method')

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$$

(Solved via Pell's 'three-column method')

Exemplum . . . non satis determinatis

$$
\begin{aligned}
5a + 3b - 2c &= 24 \\
-2a + 4b + 3c &= 5
\end{aligned}
$$

($a, b, c > 0$; found bounds for the possible values: e.g., $a < 15\frac{9}{11}$)

Linear equations — systematic practical methods

Gaussian elimination:

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Gaussian elimination:

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- Colin Maclaurin, *A treatise of algebra* (1748), §§82–85

Maclaurin on Gaussian elimination

A
TREATISE
OF
ALGEBRA,
IN
THREE PARTS.

CONTAINING

I. *The Fundamental Rules and Operations.*

II. *The Composition and Resolution of Equations of all Degrees; and the different Affections of their Roots.*

III. *The Application of* Algebra *and* Geometry *to each other.*

To which is added an

APPENDIX,

Concerning the general Properties of GEOMETRICAL LINES.

By *COLIN MACLAURIN*, M. A.
Late PROFESSOR of MATHEMATICS in the University of *Edinburgh*, and *Fellow* of the *Royal Society*.

LONDON:
Printed for A. MILLAR, and J. NOURSE, opposite to *Catherine-Street*, in the *Strand*.
M.DCC.XLVIII.

$$\begin{cases} x : y :: a : b \\ x^3 - y^3 = d \end{cases}$$

$$x = \frac{ay}{b} \text{ and } x^3 = \frac{a^3y^3}{b^3}$$

but $x^3 = d + y^3$

whence $d + y^3 = \frac{a^3y^3}{b^3}$

and $a^3y^3 - b^3y^3 = db^3$

$$y^3 = \frac{db^3}{a^3 - b^3}$$

$$y = \sqrt[3]{\frac{db^3}{a^3 - b^3}}$$

and $x = \sqrt[3]{\frac{da^3}{a^3 - b^3}}$.

DIRECTION V.

§ 82. "*If there are three unknown Quantities, there must be three Equations in order to determine them, by comparing which you may, in all Cases, find two Equations involving only two unknown Quantities; and then, by* Direct. 3d, *from these two you may deduce an Equation involving only one unknown Quantity; which may be resolved by the Rules of the last Chapter.*"

From 3 Equations involving any three unknown Quantities, *x*, *y*, and *z*, to deduce two Equations involving only two unknown Quantities, the following Rule will always serve.

RULE.

Linear equations — systematic practical methods

Gaussian elimination:

- *The nine chapters of the mathematical art*, China (c. 150 BC)
- Colin Maclaurin, *A treatise of algebra* (1748), §§82–85

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- C. F. Gauss: calculation of asteroid orbits (1810)
- from surveying, e.g., Wilhelm Jordan, *Handbuch der Vermessungskunde*, 3rd edition (1888)

Maclaurin and linear equations

EXAMPLE I.

Supp. $\begin{cases} 5x+7y=100 \\ 3x+8y=\ 80 \end{cases}$

then $y=\frac{5\times80-3\times100}{5\times8-3\times7}=\frac{100}{19}=5\frac{5}{19}$

and $x=\frac{240}{19}=12\ \frac{12}{19}$.

EXAMPLE II.

$\begin{cases} 4x+8y=\ 90 \\ 3x-2y=160 \end{cases}$

$y=\frac{4\times160-3\times90}{4\times-2-3\times8}=\frac{640-270}{-8-24}=\frac{370}{-32}=-11\frac{9}{16}$.

THEOREM II.

§ 87. Suppoſe now that there are three unknown Quantities and three Equations, then call the unknown Quantities x, y, and z.

Thus,

$\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{cases}$

Then ſhall $z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}$.

Where the Numerator conſiſts of all the different Products that can be made of three oppoſite Coefficients taken from the Orders in which z is not found; and the Denominator conſiſts of all the Products that can be made of the three op-

Colin Maclaurin, *A treatise of algebra*, 1748, p. 83

Three equations in three unknowns solved using a 'determinant-like' quantity

Maclaurin and linear equations

Chap. 12. A L G E B R A. 83

EXAMPLE I.

Supp. $\begin{cases} 5x+7y=100 \\ 3x+8y=80 \end{cases}$

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$\begin{cases} 4x+8y=90 \\ 3x-2y=160 \end{cases}$

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G 2

Colin Maclaurin, *A treatise of algebra*, 1748, p. 83

Three equations in three unknowns solved using a ‘determinant-like’ quantity

Chap. 13. A L G E B R A. 85

If four Equations are given, involving four unknown Quantities, their Values may be found much after the ſame Manner, by taking all the Products that can be made of four oppoſite Coefficients, and always prefixing contrary Signs to thoſe that involve the Products of two oppoſite Coefficients.

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Colin Maclaurin, *A treatise of algebra*, 1748, p. 83

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Notational difficulties — we run out of letters!