B01 History of Mathematics
Lecture XIV
Linear algebra
Part 2: Determinants and matrices

MT 2020 Week 7

Determinants

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Determinants

Leibniz, unpublished works, 1680s/1690s.

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Leibniz and determinants

At least as early as June 1678, Leibniz devised a new notation for coefficients, writing

$$10 + 11x + 12y = 0,$$
$$20 + 21x + 22y = 0$$

for what we would write as

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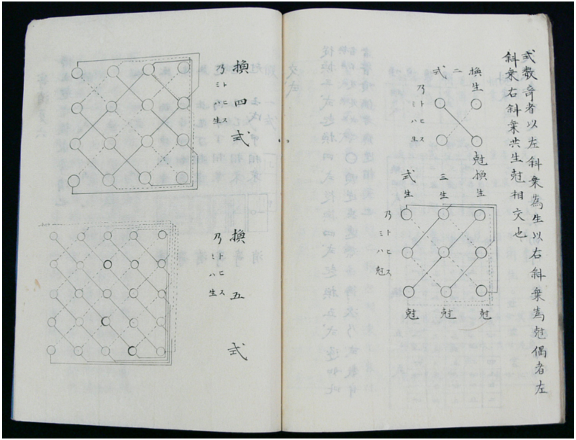
Seki and determinants

Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法 (*Method for Solving Concealed Problems*), 1683

Arranged coefficients of systems of equations in a grid, and gave schematics for construction of determinants (dotted lines indicate positive products, and solid lines negative)



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Vandermonde on elimination

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Des Équations du premier degré.

Je suppose que l'on représente par $\overset{1}{1}, \overset{2}{1}, \overset{3}{1}$, &c. $\overset{1}{2}, \overset{2}{2}, \overset{3}{2}$, &c. $\overset{1}{3}, \overset{2}{3}, \overset{3}{3}$, &c. &c. autant de différentes quantités générales, dont l'une quelconque soit $\overset{\alpha}{a}$, une autre quelconque soit $\overset{\beta}{b}$, &c. & que le produit des deux soit désigné à l'ordinaire par $\overset{\alpha}{a} \cdot \overset{\beta}{b}$.

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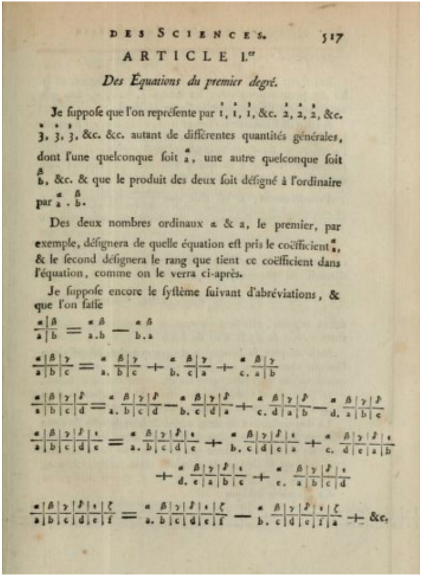

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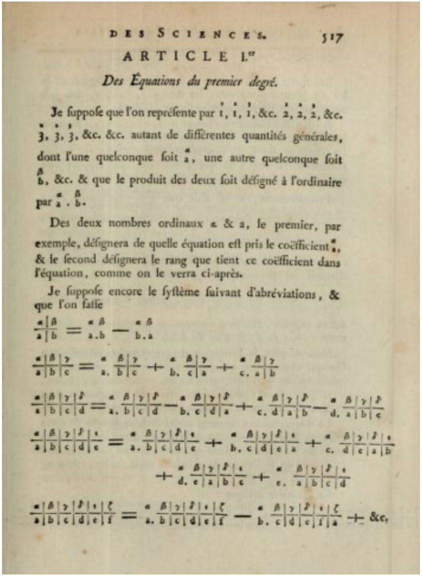

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Anachronistically, this is the determinant of the matrix:

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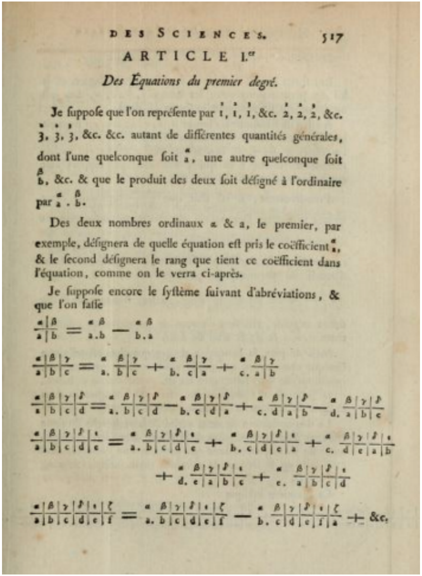

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Then continue recursively . . .

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Gauss in *Disquisitiones arithmeticae* (1801) gave the name 'determinant' to what is now called the 'discriminant' $B^2 - AC$ of the binary quadratic form $Ax^2 + 2Bxy + Cy^2$.

Cauchy on determinants

les propriétés générales des formes du second degré, c'est-à-dire des polynomes du second degré à deux ou à plusieurs variables, et il a désigné ces mêmes fonctions sous le nom de *déterminants*. Je conserverai cette dénomination qui fournit un moyen facile d'énoncer les résultats; j'observerai seulement qu'on donne aussi quelquefois aux fonctions dont il s'agit le nom de *résultantes* à deux ou à plusieurs lettres. Ainsi les deux expressions suivantes, *déterminant* et *résultante*, devront être regardées comme synonymes.

DEUXIÈME PARTIE.

DES FONCTIONS SYMÉTRIQUES ALTERNÉES DÉSIGNÉES SOUS LE NOM DE *DÉTERMINANTS*.

PREMIÈRE SECTION.

Des déterminants en général et des systèmes symétriques.

§ I^er^. Soient $a_1, a_2, \ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus que, en multipliant le produit de ces quantités ou

$$a_1 a_2 a_3 \ldots a_n$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1}),$$

on obtenait pour résultat la fonction symétrique alternée

$$S(\pm a_1 a_2^2 a_3^3 \ldots a_n^n)$$

qui, par conséquent, se trouve toujours égale au produit

$$a_1 a_2 a_3 \ldots a_n (a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit et que, dans chaque terme du développement, on remplace l'exposant de

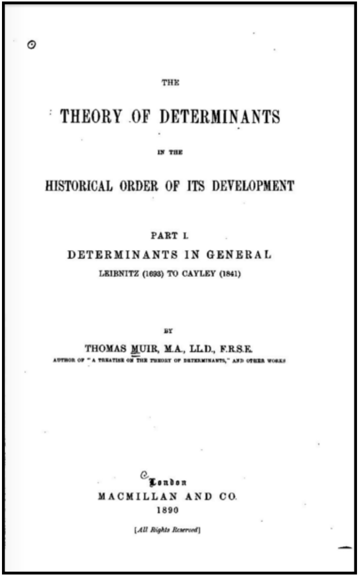
Cauchy, 'Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment', *Journal de l'École polytechnique*, 1815

Referred to Laplace, Vandermonde, Gauss, and others

Introduced the term determinant for the function of n^2 quantities (a sum of $n!$ signed products) that we now know by that name.

(See *Mathematics emerging*, §17.1.4.)

History of the theory of determinants



THE

THEORY OF DETERMINANTS

IN THE

HISTORICAL ORDER OF ITS DEVELOPMENT

PART I.

DETERMINANTS IN GENERAL

LEIBNITZ (1693) TO CAYLEY (1841)

BY

THOMAS MUIR, M.A., LL.D., F.R.S.E.

AUTHOR OF "A TREATISE ON THE THEORY OF DETERMINANTS," AND OTHER WORKS

London

MACMILLAN AND CO.

1890

Determinants were studied extensively in the 19th century.

Sir Thomas Muir, *The theory of determinants in the historical order of development* (1890–1906)

- Part I: *Determinants in general: Leibnitz (1693) to Cayley (1841)*;
- Part II: *Special determinants up to 1841*

Second edition in 4 volumes, 1906–1923; supplement, 1930.

'Eigenvalue' problems

Euler (1748): change of coordinates to reduce equation of a quadric surface $\alpha z^2 + \beta yz + \gamma xz + \delta y^2 + \epsilon xy + \zeta x^2 + \eta z + \theta y + \iota x + \chi = 0$ to its simplest form $Ap^2 + Bq^2 + Cr^2 + K = 0$ (see: *Mathematics emerging*, §17.2.1.)

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Laplace (1787): symmetry of coefficients in a set of linear differential equations leads to real ‘eigenvalues’ (see: *Mathematics emerging*, §17.2.2.)

Cauchy (1829): a symmetric matrix is diagonalisable by a real orthogonal change of variables (see: *Mathematics emerging*, §17.2.3.)

Matrices and their determinants

Gauss, *Disquisitiones arithmeticae* (1801): transformation of quadratic forms $ax^2 + 2bxy + cy^2$ by change of variables

$$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y'$$

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Moreover, the ‘determinants’ (our sense) multiply.

NB. All Gauss’ coefficients were integers

(See *Mathematics emerging*, §17.3.1.)

Early origins of matrices

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> *For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding of pth order.*

Early origins of matrices

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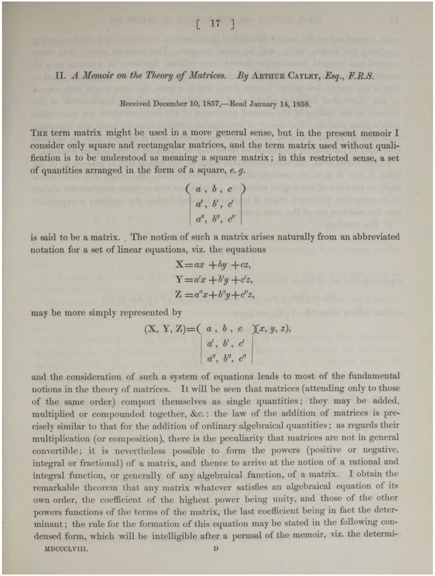
But he did not operate with matrices

The definition of matrices



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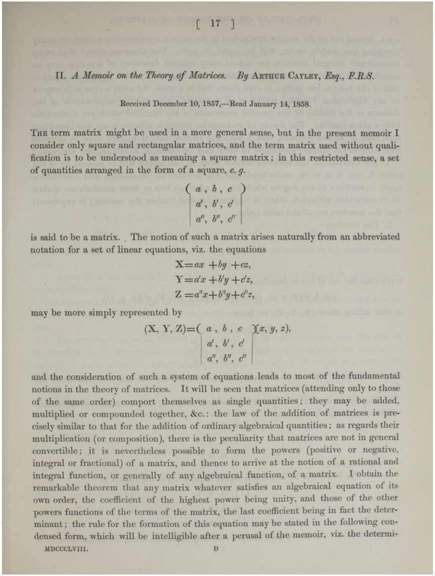

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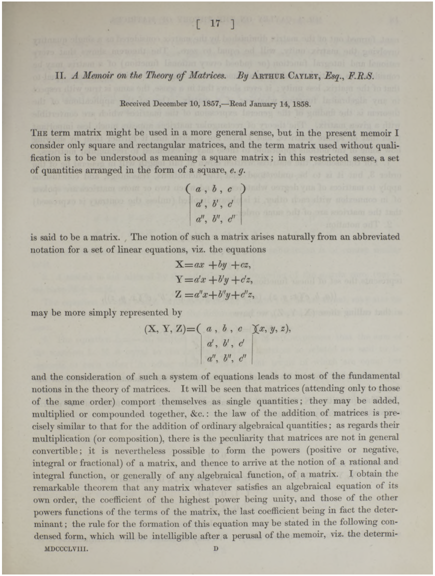

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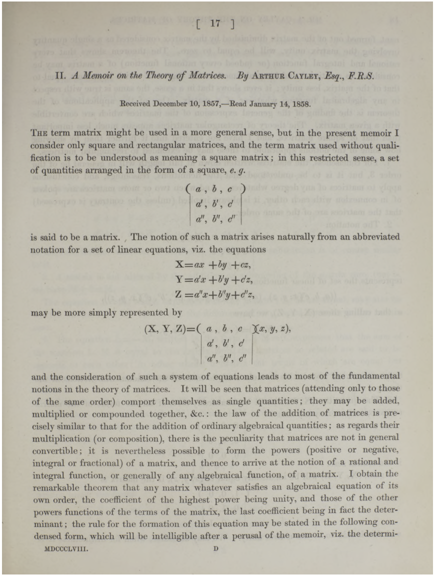

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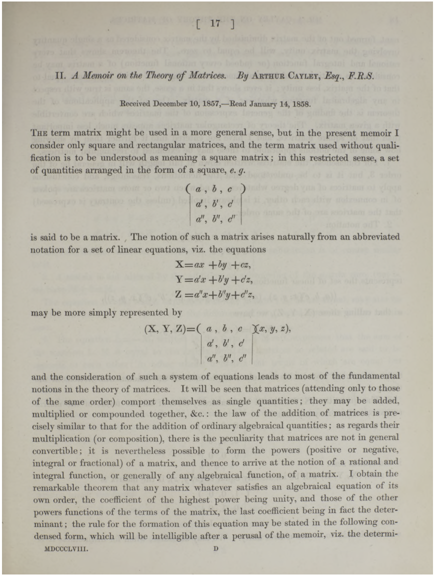

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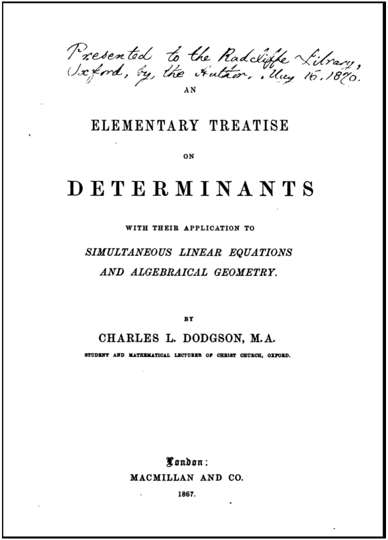
(See *Mathematics emerging*, §17.3.2.)

Determinants persist



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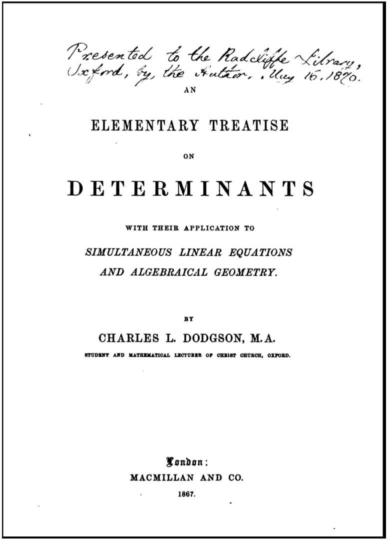
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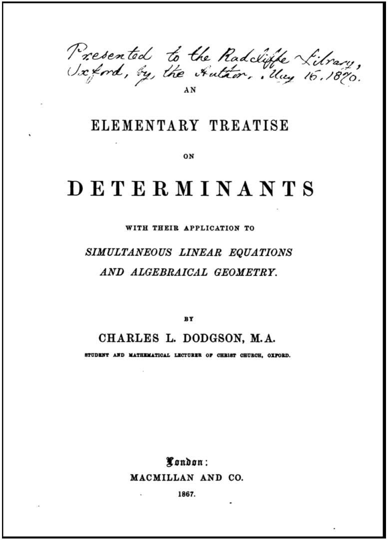

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Proposed $i \rceil j$ instead

Matrices elsewhere

Matrix algebra appears in Hamilton's *Lectures on Quaternions* (1853) as 'linear and vector functions' (including his version of the Cayley–Hamilton Theorem, stated and proved in terms of quaternions)

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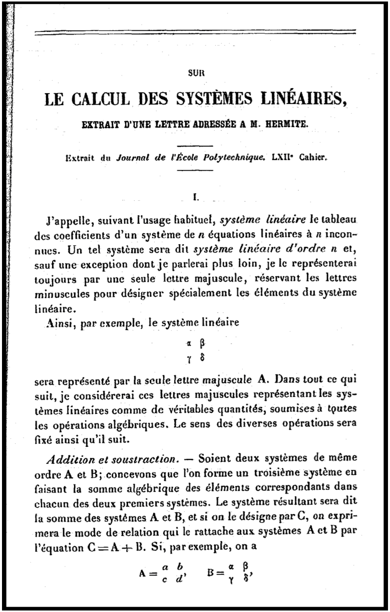
Matrices were also devised by Laguerre in his paper 'Sur le calcul des systèmes linéaires' (*J. École polytechnique*, 1867)



SUR

LE CALCUL DES SYSTÈMES LINÉAIRES,

EXTRAIT D'UNE LETTRE ADRESSÉE A M. HERMITE.

Extrait du *Journal de l'École Polytechnique*, LXII[e] Cahier.

I.

J'appelle, suivant l'usage habituel, *système linéaire* le tableau des coefficients d'un système de *n* équations linéaires à *n* inconnues. Un tel système sera dit *système linéaire d'ordre n* et, sauf une exception dont je parlerai plus loin, je le représenterai toujours par une seule lettre majuscule, réservant les lettres minuscules pour désigner spécialement les éléments du système linéaire.

Ainsi, par exemple, le système linéaire

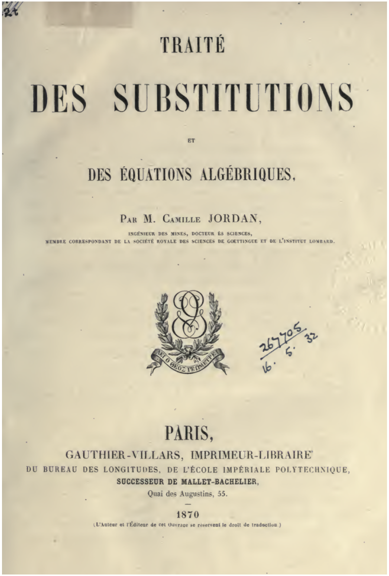
$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$$

sera représenté par la seule lettre majuscule A. Dans tout ce qui suit, je considérerai ces lettres majuscules représentant les systèmes linéaires comme de véritables quantités, soumises à toutes les opérations algébriques. Le sens des diverses opérations sera fixé ainsi qu'il suit.

Addition et soustraction. — Soient deux systèmes de même ordre A et B; concevons que l'on forme un troisième système en faisant la somme algébrique des éléments correspondants dans chacun des deux premiers systèmes. Le système résultant sera dit la somme des systèmes A et B, et si on le désigne par C, on exprimera le mode de relation qui le rattache aux systèmes A et B par l'équation $C = A + B$. Si, par exemple, on a

$$A = \begin{matrix} a & b \\ c & d \end{matrix}, \qquad B = \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix},$$

Jordan and linear substitutions



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ET

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MEMBRE CORRESPONDANT DE LA SOCIÉTÉ ROYALE DES SCIENCES DE GOETTINGUE ET DE L'INSTITUT LOMBARD.

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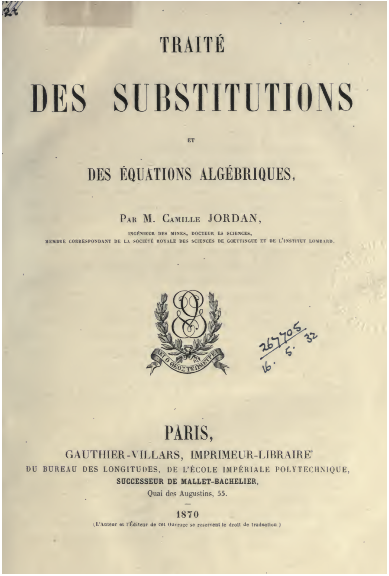

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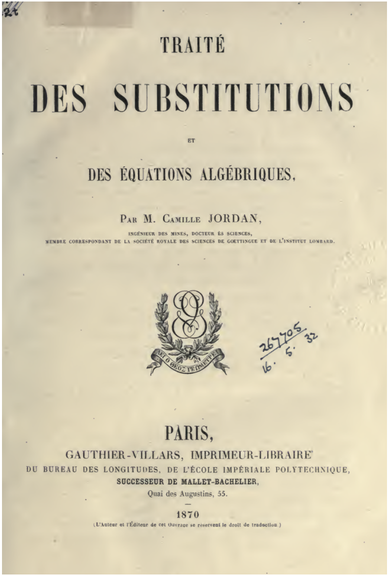

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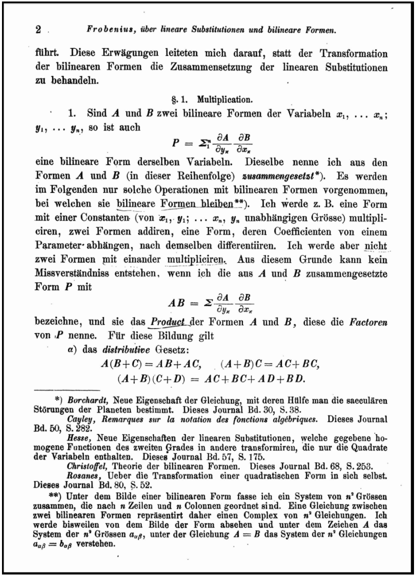
- developed his ideas to 'Jordan canonical form' for complex matrices in his studies 1872–4 of linear differential equations

German contributions



2 *Frobenius, über lineare Substitutionen und bilineare Formen.*

führt. Diese Erwägungen leiteten mich darauf, statt der Transformation der bilinearen Formen die Zusammensetzung der linearen Substitutionen zu behandeln.

§. 1. Multiplication.

1. Sind A und B zwei bilineare Formen der Variabeln $x_1, \dots x_n$; $y_1, \dots y_n$, so ist auch

$$P = \Sigma_1^n \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

eine bilineare Form derselben Variabeln. Dieselbe nenne ich aus den Formen A und B (in dieser Reihenfolge) *zusammengesetzt**). Es werden im Folgenden nur solche Operationen mit bilinearen Formen vorgenommen, bei welchen sie bilineare Formen bleiben**). Ich werde z. B. eine Form mit einer Constanten (von $x_1, y_1; \dots x_n, y_n$ unabhängigen Grösse) multipliciren, zwei Formen addiren, eine Form, deren Coefficienten von einem Parameter abhängen, nach demselben differentiiren. Ich werde aber nicht zwei Formen mit einander multipliciren. Aus diesem Grunde kann kein Missverständniss entstehen, wenn ich die aus A und B zusammengesetzte Form P mit

$$AB = \Sigma \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

bezeichne, und sie das *Product* der Formen A und B, diese die *Factoren* von P nenne. Für diese Bildung gilt

α) das *distributive* Gesetz:

$$A(B+C) = AB+AC, \quad (A+B)C = AC+BC,$$
$$(A+B)(C+D) = AC+BC+AD+BD.$$

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(See *Mathematics emerging*, §17.3.3.)

German contributions

2 *Frobenius, über lineare Substitutionen und bilineare Formen.*

führt. Diese Erwägungen leiteten mich darauf, statt der Transformation der bilinearen Formen die Zusammensetzung der linearen Substitutionen zu behandeln.

§. 1. Multiplication.

1. Sind A und B zwei bilineare Formen der Variabeln $x_1, \dots x_n$; $y_1, \dots y_n$, so ist auch

$$P = \Sigma_1^n \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

eine bilineare Form derselben Variabeln. Dieselbe nenne ich aus den Formen A und B (in dieser Reihenfolge) *zusammengesetzt**). Es werden im Folgenden nur solche Operationen mit bilinearen Formen vorgenommen, bei welchen sie bilineare Formen bleiben**). Ich werde z. B. eine Form mit einer Constanten (von $x_1, y_1; \dots x_n, y_n$ unabhängigen Grösse) multipliciren, zwei Formen addiren, eine Form, deren Coefficienten von einem Parameter abhängen, nach demselben differentiiren. Ich werde aber nicht zwei Formen mit einander multipliciren. Aus diesem Grunde kann kein Missverständniss entstehen, wenn ich die aus A und B zusammengesetzte Form P mit

$$AB = \Sigma \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

bezeichne, und sie das *Product* der Formen A und B, diese die *Factoren* von P nenne. Für diese Bildung gilt

α) das *distributive* Gesetz:

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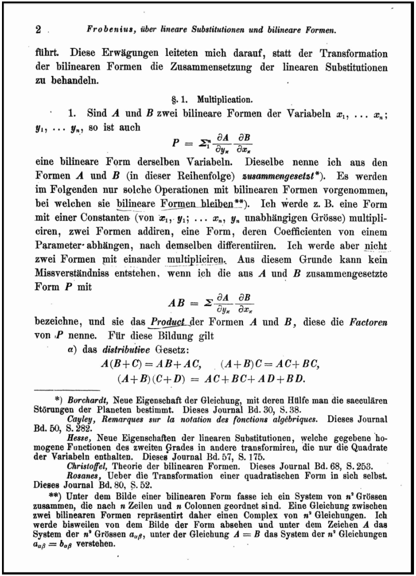

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A recommended secondary source: Thomas Hawkins, 'Another look at Cayley and the theory of matrices', *Archives internationales d'histoire des sciences* **26** (1977), 82–112