BO1 History of Mathematics
Lecture XV
Geometry and number theory
Part 1: Non-Euclidean geometry

MT 2020 Week 8

Summary

Part 1

- Euclid's *Elements* revisited
- The parallel postulate
- Non-Euclidean geometry

Part 2

- Number theory down the centuries

Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

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Book VI:	similarity, proportion
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Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

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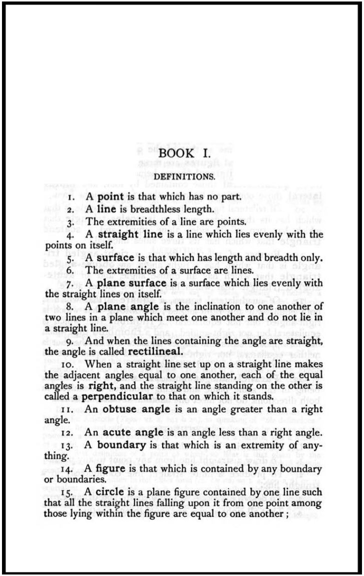
Euclid in English



BOOK I.

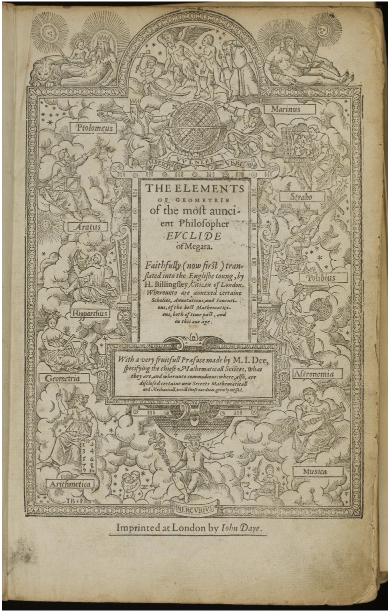
DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal**.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

Canonical English edition by Sir Thomas L. Heath, 1908

See also the Reading Euclid Project

Billingsley's Euclid, 1570



The Elements of Geometrie:

"Faithfully (now first) translated into the Englishe toung" by H. Billingsley, London, 1570

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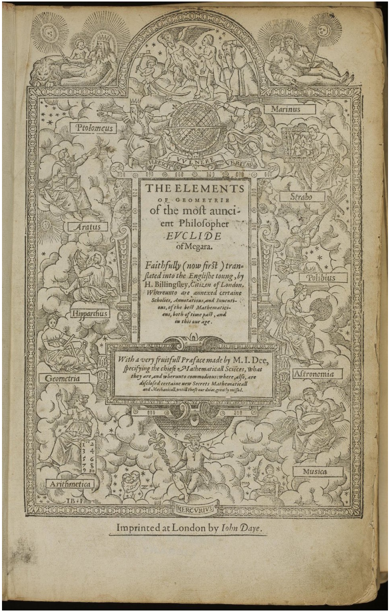

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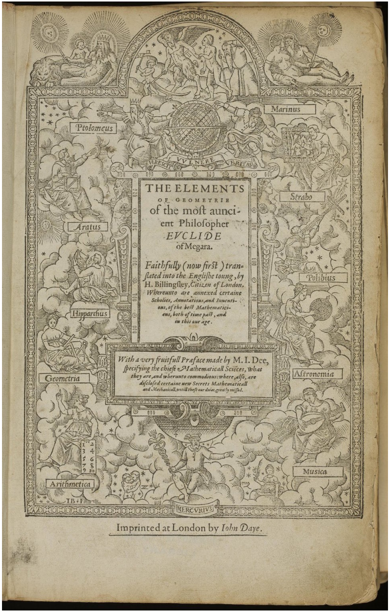

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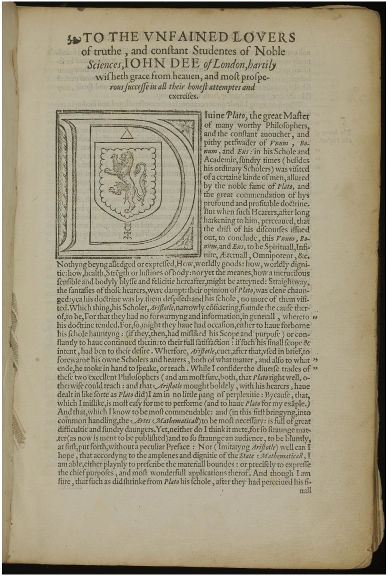
Preface by John Dee

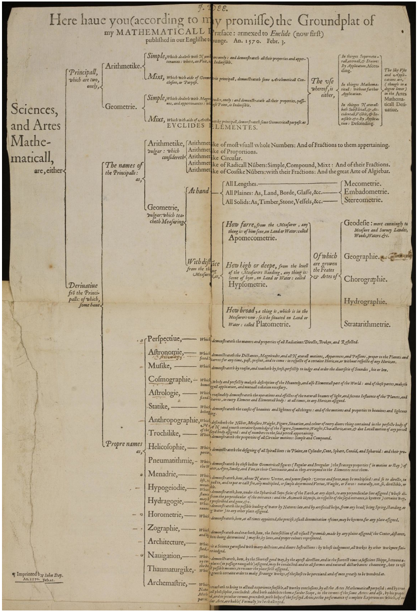
Dee's Preface



TO THE VNFAINED LOVERS of truthe, and conſtant Studentes of Noble *Sciences,* IOHN DEE *of London, hartily* wiſheth grace from heauen, and moſt proſpe*rous ſucceſſe in all their honeſt attemptes and* exerciſes.

Diuine *Plato,* the great Maſter of many worthy Philoſophers, and the conſtant auoucher, and pithy perſwader of *Vnum, Bonum,* and *Ens:* in his Schole and Academie, ſundry times (beſides his ordinary Scholers) was viſited of a certaine kinde of men, allured by the noble fame of *Plato,* and the great commendation of hys profound and profitable doctrine. But when ſuch Hearers, after long harkening to him, perceaued, that the drift of his diſcourſes iſſued out, to conclude, this *Vnum, Bonum,* and *Ens,* to be Spirituall, Infinite, Æternall, Omnipotent, &c. Nothyng beyng alledged or expreſſed, How, worldly goods: how, worldly dignitie: how, health, Strẽgth or luſtines of body: nor yet the meanes, how a merueilous ſenſible and bodyly blyſſe and felicitie hereafter, might be atteyned: Straightway, the fantaſies of thoſe hearers, were dampt: their opinion of *Plato,* was clene chaunged: yea his doctrine was by them deſpiſed: and his ſchole, no more of them viſited. Which thing, his Scholer, *Ariſtotle,* narrowly cõſidering, founde the cauſe therof, to be, For that they had no forwarnyng and information, in generall, whereto „ his doctrine tended. For, ſo, might they haue had occaſion, either to haue forborne his ſchole hauntyng: (if they, then, had miſliked his Scope and purpoſe) or conſtantly to haue continued therin: to their full ſatiſfaction: if ſuch his finall ſcope & intent, had ben to their deſire. Wherfore, *Ariſtotle,* euer, after that, vſed in brief, to forewarne his owne Scholers and hearers, both of what matter, and alſo to what „ ende, he tooke in hand to ſpeake, or teach. While I conſider the diuerſe trades of „ theſe two excellent Philoſophers (and am moſt ſure, both, that *Plato* right well, otherwiſe could teach: and that *Ariſtotle* mought boldely, with his hearers, haue dealt in like ſorte as *Plato* did) I am in no little pang of perplexitie: Bycauſe, that, which I miſlike, is moſt eaſy for me to performe (and to haue *Plato* for my exãple.) And that, which I know to be moſt commendable: and (in this firſt bringyng, into common handling, the *Artes Mathematicall*) to be moſt neceſſary: is full of great difficultie and ſundry daungers. Yet, neither do I think it mete, for ſo ſtraunge matter (as now is ment to be publiſhed) and to ſo ſtraunge an audience, to be bluntly, at firſt, put forth, without a peculiar Preface: Nor (Imitatyng *Ariſtotle*) well can I hope, that accordyng to the amplenes and dignitie of the *State Mathematicall,* I am able, either playnly to preſcribe the materiall boundes: or preciſely to expreſſe the chief purpoſes, and moſt wonderfull applications therof. And though I am ſure, that ſuch as did ſhrinke from *Plato* his ſchole, after they had perceiued his finall

Dee's 'Groundplat'



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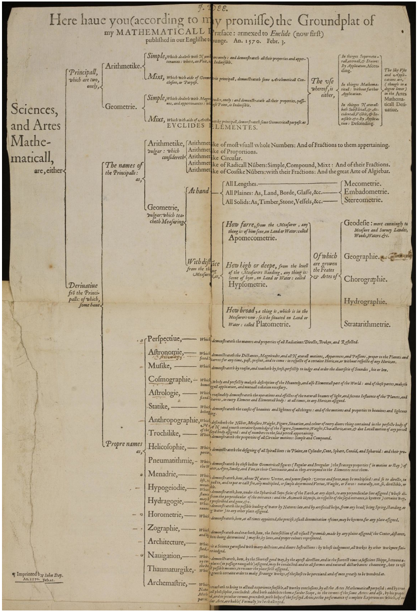
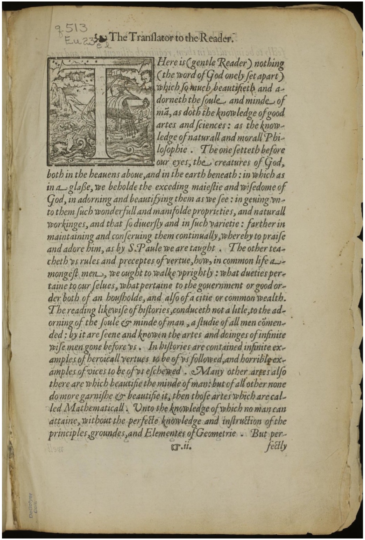


See: Jennifer M. Rampling, 'The Elizabethan mathematics of everything: John Dee's 'Mathematicall praeface' to Euclid's *Elements*', *BSHM Bulletin: Journal of the British Society for the History of Mathematics* **26**(3) (2011) 135–146

Billingsley's Preface, pp. 1, 3



THere is (gentle Reader) nothing (the word of God onely ſet apart) which ſo much beautifieth and adorneth the ſoule and minde of mã, as doth the knowledge of good artes and ſciences: as the knowledge of naturall and morall Philoſophie. The one ſetteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glaſſe, we beholde the exceding maieſtie and wiſedome of God, in adorning and beautifying them as we ſee: in geuing vnto them ſuch wonderfull and manifolde proprieties, and naturall workinges, and that ſo diuerſly and in ſuch varietie: farther in maintaining and conſeruing them continually, whereby to praiſe and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongeſt men, we ought to walke vprightly: what dueties pertaine to our ſelues, what pertaine to the gouernment or good order both of an houſholde, and alſo of a citie or common wealth. The reading likewiſe of hiſtories, conduceth not a litle, to the adorning of the ſoule & minde of man, a ſtudie of all men cõmended: by it are ſeene and knowen the artes and doinges of infinite wiſe men gone before vs. In hiſtories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eſchewed. Many other artes alſo there are which beautifie the minde of man: but of all other none do more garniſhe & beautifie it, then thoſe artes which are called Mathematicall. Unto the knowledge of which no man can attaine, without the perfecte knowledge and inſtruction of the principles, groundes, and Elementes of Geometrie. But perfectly

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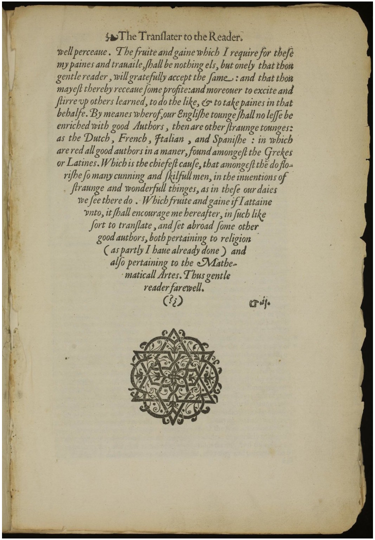

well perceaue. The fruite and gaine which I require for theſe my paines and trauaile, ſhall be nothing els, but onely that thou gentle reader, will gratefully accept the ſame: and that thou mayeſt thereby receaue ſome profite: and moreouer to excite and ſtirre vp others learned, to do the like, & to take paines in that behalfe. By meanes wherof, our Engliſhe tounge ſhall no leſſe be enriched with good Authors, then are other ſtraunge tounges: as the Dutch, French, Italian, and Spaniſhe: in which are red all good authors in a maner, found amongeſt the Grekes or Latines. Which is the chiefeſt cauſe, that amongeſt thẽ do floriſhe ſo many cunning and ſkilfull men, in the inuentions of ſtraunge and wonderfull thinges, as in theſe our daies we ſee there do. Which fruite and gaine if I attaine vnto, it ſhall encourage me hereafter, in ſuch like ſort to tranſlate, and ſet abroad ſome other good authors, both pertaining to religion (as partly I haue already done) and alſo pertaining to the Mathematicall Artes. Thus gentle reader farewell.

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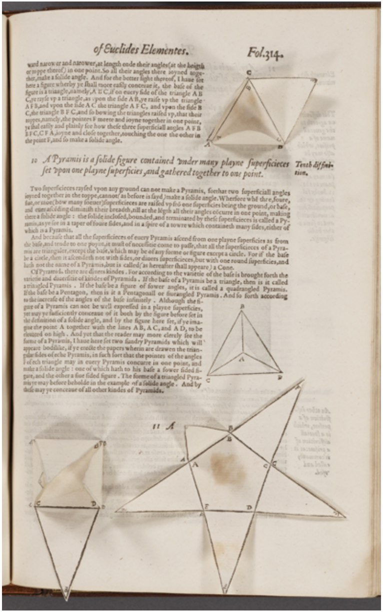
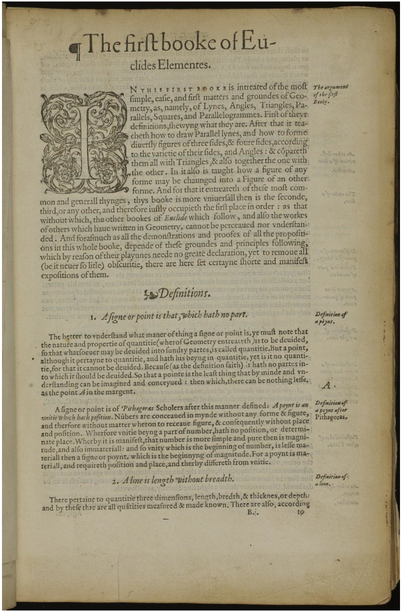
Pop-up Euclid



Book I: definitions



¶ The first booke of Euclides Elementes.

The argumentes of the first Booke.

IN THIS FIRST BOOKE is intreated of the most simple, easie, and first matters and groundes of Geometry, as, namely, of Lynes, Angles, Triangles, Parallels, Squares, and Parallelogrammes. First of theyr definitions, shewyng what they are. After that it teacheth how to draw Parallel lynes, and how to forme diuersly figures of three sides, & foure sides, according to the varietie of their sides, and Angles: & cōpareth them all with Triangles, & also together the one with the other. In it also is taught how a figure of any forme may be chaunged into a Figure of an other forme. And for that it entreateth of these most common and generall thynges, thys booke is more vniuersall then is the seconde, third, or any other, and therefore iustly occupieth the first place in order: as that without which, the other bookes of *Euclide* which follow, and also the workes of others which haue written in Geometry, cannot be perceaued nor vnderstanded. And forasmuch as all the demonstrations and proofes of all the propositions in this whole booke, depende of these groundes and principles following, which by reason of their playnnes neede no greate declaration, yet to remoue all (be it neuer so litle) obscuritie, there are here set certayne shorte and manifest expositions of them.

Definitions.

Definition of a poynt.

1. *A signe or point is that, which hath no part.*

The better to vnderstand what maner of thing a signe or point is, ye must note that the nature and propertie of quantitie (wherof Geometry entreateth) is to be deuided, so that whatsoeuer may be deuided into sundry partes, is called quantitie. But a point, although it pertayne to quantitie, and hath his beyng in quantitie, yet is it no quantitie, for that it cannot be deuided. Because (as the definition saith) it hath no partes into which it should be deuided. So that a pointe is the least thing that by minde and vnderstanding can be imagined and conceyued: then which, there can be nothing lesse, as the point *A* in the margent. A.

Definition of a poynt after Pithagoras.

A signe or point is of *Pithagoras* Scholers after this manner defined: *A poynt is an vnitie which hath position.* Nūbers are conceaued in mynde without any forme & figure, and therfore without matter wheron to receaue figure, & consequently without place and position. Wherfore vnitie beyng a part of number, hath no position, or determinate place. Wherby it is manifest, that number is more simple and pure then is magnitude, and also immateriall: and so vnity which is the beginning of number, is lesse materiall then a signe or poynt, which is the beginnyng of magnitude. For a poynt is materiall, and requireth position and place, and therby differeth from vnitie.

Definition of a line.

2. *A line is length without breadth.*

There pertaine to quantitie three dimensions, length, bredth, & thicknes, or depth: and by these thre are all quātities measured & made known. There are also, according

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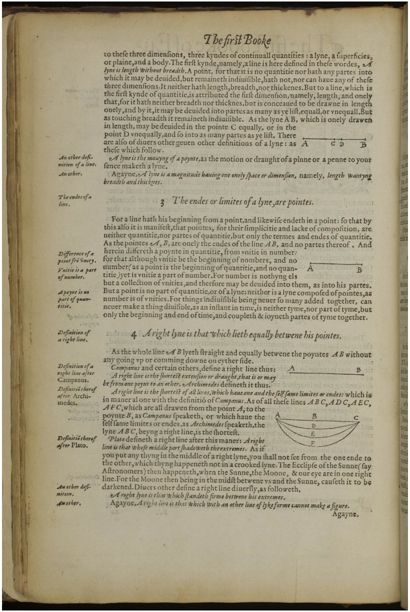


to these three dimensions, three kyndes of continuall quantities: a lyne, a superficies, or plaine, and a body. The first kynde, namely, a line is here defined in these wordes, *A lyne is length without breadth.* A point, for that it is no quantitie nor hath any partes into which it may be deuided, but remaineth indiuisible, hath not, nor can haue any of these three dimensions. It neither hath length, breadth, nor thickenes. But to a line, which is the first kynde of quantitie, is attributed the first dimension, namely, length, and onely that, for it hath neither breadth nor thicknes, but is conceaued to be drawne in length onely, and by it, it may be deuided into partes as many as ye list, equall, or vnequall. But as touching breadth it remaineth indiuisible. As the lyne A B, which is onely drawen in length, may be deuided in the pointe C equally, or in the point D vnequally, and so into as many partes as ye list. There are also of diuers other geuen other definitions of a lyne: as these which follow.

An other definition of a line.

A lyne is the mouyng of a poynte, as the motion or draught of a pinne or a penne to your sence maketh a lyne.

An other.

Agayne, *A lyne is a magnitude hauing one onely space or dimension,* namely, *length wantyng breadth and thicknes.*

The endes of a line.

3 *The endes or limites of a lyne, are pointes.*

For a line hath his beginning from a point, and likewise endeth in a point: so that by this also it is manifest, that pointes, for their simplicitie and lacke of composition, are neither quantitie, nor partes of quantitie, but only the termes and endes of quantitie. As the pointes *A*, *B*, are onely the endes of the line *A B*, and no partes thereof. And herein differeth a poynte in quantitie, from vnitie in number: for that although vnitie be the beginning of nombers, and no number (as a point is the beginning of quantitie, and no quantitie) yet is vnitie a part of number. For number is nothyng els but a collection of vnities, and therfore may be deuided into them, as into his partes. But a point is no part of quantitie, or of a lyne: neither is a lyne composed of pointes, as number is of vnities. For things indiuisible being neuer so many added together, can neuer make a thing diuisible, as an instant in time, is neither tyme, nor part of tyme, but only the beginning and end of time, and coupleth & ioyneth partes of tyme together.

Difference of a point frō Vnity.

Vnitie is a part of number.

A poynt is no part of quantitie.

Definition of a right line.

4 *A right lyne is that which lieth equally betwene his pointes.*

As the whole line *A B* lyeth straight and equally betwene the poyntes *A B* without any going vp or comming downe on eyther side.

Definition of a right line after Campanus.

Campanus and certain others, define a right line thus: *A right line is the shortest extension or draught, that is or may be from one poynt to an other.* *Archimedes* defineth it thus.

Definitiō therof after Archimedes.

A right line is the shortest of all lines, which haue one and the selfsame limites or endes: which is in maner al one with the definitiō of *Campanus*. As of all these lines *A B C*, *A D C*, *A E C*, *A F C*, which are all drawen from the point *A*, to the poynte *B*, as *Campanus* speaketh, or which haue the selfsame limites or endes, as *Archimedes* speaketh, the lyne *A B C*, beyng a right line, is the shortest.

Definitiō therof after Plato.

Plato defineth a right line after this maner: *A right line is that whose middle part shadoweth the extremes.* As if you put any thyng in the middle of a right lyne, you shall not see from the one ende to the other, which thyng happeneth not in a crooked lyne. The Ecclipse of the Sunne (say Astronomers) then happeneth, when the Sunne, the Moone, & our eye are in one right line. For the Moone then being in the midst betwene vs and the Sunne, causeth it to be darkened. Diuers other define a right line diuersly, as followeth.

An other definition.

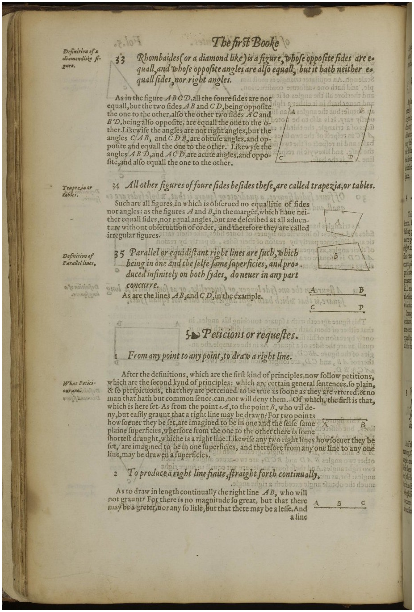
A right lyne is that which standeth firme betwene his extremes.

Another.

Agayne, *A right line is that which with an other line of lyke forme cannot make a figure.*

Agayne,

Book I: postulates



Definition of a diamondlike figure.

33 *Rhombaides (or a diamond like) is a figure, whoſe oppoſite ſides are equall, and whoſe oppoſite angles are alſo equall, but it hath neither equall ſides, nor right angles.*

As in the figure ABCD, all the foure ſides are not equall, but the two ſides AB and CD, being oppoſite the one to the other, alſo the other two ſides AC and BD, being alſo oppoſite, are equall the one to the other. Likewiſe the angles are not right angles, but the angles CAB, and CDB, are obtuſe angles, and oppoſite and equall the one to the other. Likewyſe the angles ABD, and ACD, are acute angles, and oppoſite, and alſo equall the one to the other.

Trapezia or tables.

34 *All other figures of foure ſides beſides theſe, are called trapezia, or tables.*

Such are all figures, in which is obſerued no equallitie of ſides nor angles: as the figures A and B, in the margēt, which haue neither equall ſides, nor equal angles, but are deſcribed at all aduenture without obſeruation of order, and therefore they are called irregular figures.

Definition of Parallel lines.

35 *Parallel or equidiſtant right lines are ſuch, which being in one and the ſelfe ſame ſuperficies, and produced infinitely on both ſydes, do neuer in any part concurre.*

As are the lines AB, and CD, in the example.

Peticions or requeſtes.

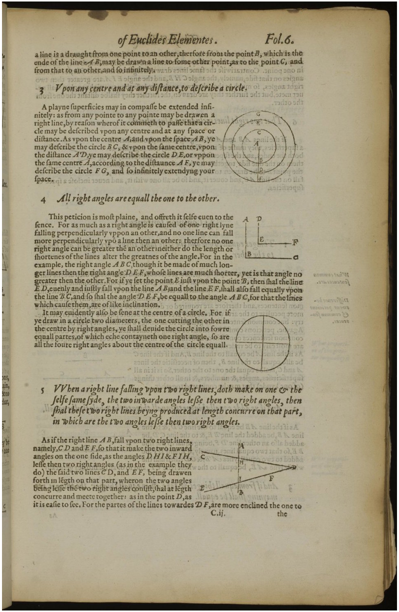
1 *From any point to any point, to draw a right line.*

What Peticions are.

After the definitions, which are the firſt kind of principles, now follow petitions, which are the ſecond kynd of principles: which are certain general ſentences, ſo plain, & ſo perſpicuous, that they are perceiued to be true as ſoone as they are vttered, & no man that hath but common ſence, can, nor will deny them. Of which, the firſt is that, which is here ſet. As from the point A, to the point B, who wil deny, but eaſily graunt that a right line may be drawn? For two points howſoeuer they be ſet, are imagined to be in one and the ſelfe ſame plaine ſuperficies, wherfore from the one to the other there is ſome ſhorteſt draught, whiche is a right line. Likewiſe any two right lines howſoeuer they be ſet, are imagined to be in one ſuperficies, and therefore from any one line to any one line, may be drawen a ſuperficies.

2 *To produce a right line finite, ſtraight forth continually.*

As to draw in length continually the right line AB, who will not graunt? For there is no magnitude ſo great, but that there may be a greter, nor any ſo litle, but that there may be a leſſe. And a line



a line is a draught from one point to an other, therfore from the point B, which is the ende of the line AB, may be drawn a line to ſome other point, as to the point C, and from that to an other, and ſo infinitely.

3 *Vpon any centre and at any diſtance, to deſcribe a circle.*

A playne ſuperficies may in compaſſe be extended infinitely: as from any pointe to any pointe may be drawen a right line, by reaſon wherof it commeth to paſſe that a circle may be deſcribed vpon any centre and at any ſpace or diſtance. As vpon the centre A, and vpon the ſpace AB, ye may deſcribe the circle BC, & vpon the ſame centre, vpon the diſtance AD, ye may deſcribe the circle DE, or vppon the ſame centre A, according to the diſtaunce AF, ye may deſcribe the circle FG, and ſo infinitely extendyng your ſpace.

4 *All right angles are equall the one to the other.*

This peticion is moſt plaine, and offreth it ſelfe euen to the ſence. For as much as a right angle is cauſed of one right lyne falling perpendicularly vppon an other, and no one line can fall more perpendicularly vpō a line then an other: therfore no one right angle can be greater thē an other: neither do the length or ſhortenes of the lines alter the greatnes of the angle. For in the example, the right angle ABC, though it be made of much longer lines then the right angle DEF, whoſe lines are much ſhorter, yet is that angle no greater then the other. For if ye ſet the point E iuſt vpon the point B, then ſhal the line ED, euenly and iuſtly fall vpon the line AB, and the line EF, ſhall alſo fall equally vpon the line BC, and ſo ſhal the angle DEF, be equall to the angle ABC, for that the lines which cauſe them, are of like inclination.

It may euidently alſo be ſene at the centre of a circle. For if ye draw in a circle two diameters, the one cutting the other in the centre by right angles, ye ſhall deuide the circle into fowre equall partes, of which eche contayneth one right angle, ſo are all the foure right angles about the centre of the circle equall.

5 *When a right line falling vpon two right lines, doth make on one & the ſelfe ſame ſyde, the two inwarde angles leſſe then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſſe then two right angles.*

As if the right line AB, fall vpon two right lines, namely, CD and EF, ſo that it make the two inward angles on the one ſide, as the angles DHI & FIH, leſſe then two right angles (as in the example they do) the ſaid two lines CD, and EF, being drawen forth in lēgth on that part, wheron the two angles being leſſe thē two right angles conſiſt, ſhal at lēgth concurre and meete together: as in the point D, as it is eaſie to ſee. For the partes of the lines towardes DF, are more enclined the one to the

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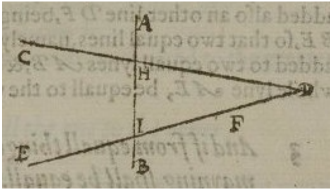

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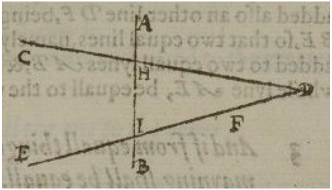

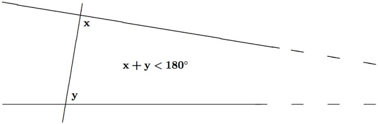

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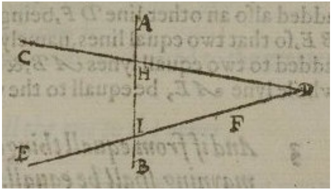

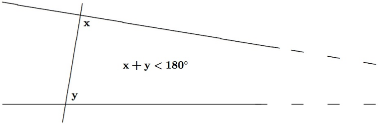

Equivalent formulation (Proclus, 5th century; John Playfair, 1795): given a straight line L and a point P not on L there is one and only one straight line through P that is parallel to L.

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See Heath, pp. 202–220

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Al-Tusi's thoughts found their way into Europe via the writings (1298) of his son Sadr al-Tusi

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Similar results derived by Johann Heinrich Lambert (1728–1777) in his *Theorie der Parallellinien* (1766)

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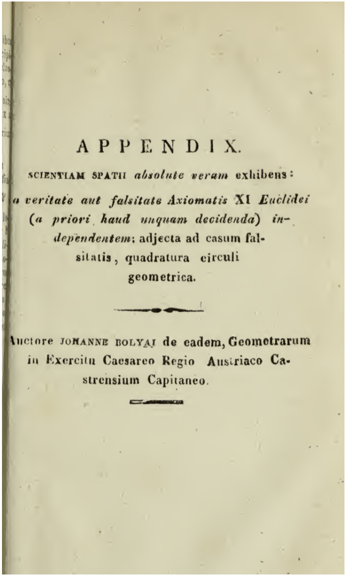
Pursued (against paternal advice) and solved by János Bolyai (1802–1860): "I have created a new and different world out of nothing" (1823)

Bolyai's geometry



APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:
a veritate aut falsitate Axiomatis XI *Euclidei* (*a priori haud unquam decidenda*) *independentem*; adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

Published as appendix 'The science absolute of space: independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)' to father's textbook *Tentamen iuventutem studiosam in elementa matheosos introducendi* (1832)

English translation by George Bruce Halstead (1896)

Meanwhile in Russia...

Non-Euclidean geometry developed independently by Nikolai Ivanovich Lobachevskii [Николай Иванович Лобачевский] (1792–1856) using the negation of Playfair's axiom

Lobachevskii's works

Geometrische Untersuchungen

zur

Theorie der Parallellinien

von

Nicolaus Lobatschewsky,

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