Michaelmas 2020 **B8.5 Graph Theory** **03/10/20**

Definitions, Notation and Terminology

Notational Conventions

$[n] = \{1, 2, \ldots, n\}$.

$\binom{X}{k} = \{A \subseteq X : |A| = k\}$: set of k-element subsets of X. Some authors write $X^{(k)}$.

Graphs

A *graph* is an ordered pair (V, E) where V is a non-empty finite set and $E \subseteq \binom{V}{2}$.

The *vertex set* is $V = V(G)$ and *edge set* is $E = E(G)$.

The *order* $|G|$ of G is $|V|$ – the number of vertices.

The *size* $e(G)$ of G is $|E|$ – the number of edges.

For vertices $u \neq v$: $uv = \{u, v\} = vu$.

The *endvertices* or *ends* of an edge uv are u and v.

Vertices u, v are *adjacent* if $uv \in E$,

A vertex v and edge e are *incident* if v is an endvertex of e,

Edges e and f *meet* if they share a vertex.

The *neighbourhood* of v is $N(v) = N_G(v) = \{u : uv \in E\}$.

The *degree* of v is $d(v) = d_G(v) = |N(v)|$.

A vertex v is *isolated* if $d(v) = 0$.

A vertex v is a *leaf* if $d(v) = 1$.

Isomorphism

An *isomorphism* from a graph G to a graph H is a bijection $\phi\colon V(G) \to V(H)$ such that $\phi(v)\phi(w) \in E(H)$ iff $vw \in E(G)$.

G and H are *isomorphic* if such a ϕ exists.

Subgraphs

A graph H is a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If $W \subseteq V(G)$ then $G[W]$, the subgraph *induced* by W, is $(W, E(G) \cap \binom{W}{2})$, the graph formed by W and all edges of G with ends in W.

An *induced subgraph* of G is any such subgraph $G[W]$.

H is a *spanning subgraph* of G if $H \subseteq G$ and $V(H) = V(G)$.

Operations on graphs

The *complement* of $G = (V, E)$ is $\overline{G} = (V, \binom{V}{2} \setminus E)$.

A *non-edge* of G is an edge of $\overline{G}$.

For $e \in E(G)$, the graph obtained by deleting e is $G - e = (V, E \setminus \{e\})$.

For $e \in E(\overline{G})$, the graph obtained by adding e is $G + e = (V, E \cup \{e\})$.

For $v \in V$, define $G - v = G[V \setminus \{v\}]$, i.e., delete v and any incident edges.

The *union* of $G = (V, E)$ and $H = (V', E')$ is $G \cup H = (V \cup V, E \cup E')$. The union is *edge (vertex) disjoint* if the two edge (vertex) sets are disjoint.

Standard graphs

K_n: complete graph on $n \geqslant 1$ vertices $= ([n], \binom{[n]}{2})$.

E_n: empty graph on $n \geqslant 1$ vertices $= ([n], \emptyset)$.

P_n: path on $n \geqslant 1$ vertices ($n - 1$ edges) $= ([n], \{12, 23, \dots, (n-1)n\})$.

C_n: cycle on $n \geqslant 3$ vertices (also n edges) $= ([n], \{12, 23, \dots, (n-1)n, n1)$.

$K_{a,b}$: complete bipartite graph with a vertices in one part and b in the other.

$K_r(t)$: complete r-partite graph with t vertices in each of the r partite classes.

Further definitions

A graph G is *connected* if any two vertices are joined by a path/walk.

The *components* of G are the maximal connected subgraphs.

A *bridge* in G is an edge e whose deletion would disconnect the component of G containing e.

A *cut vertex* in G is a vertex v whose deletion would disconnect the component of G containing v.

A graph is *acyclic* if it has no subgraph that is a cycle (i.e., is isomorphic to some C_n).

A *tree* is a connected acyclic graph.

A *forest* is an acyclic graph.

A graph G is *bipartite* (*r-partite*) if we can partition the vertex set into 2 (r) disjoint sets $X_1, \dots, X_r$ so that every edge is of the form uv, $u \in X_i$, $v \in X_j$ with $i \neq j$.

If v is a vertex of $G = (V, E)$ and A and B are disjoint subsets of V we write

$N_A(v) = A \cap N(v)$ for the *neighbourhood of* v *in* A,

$d_A(v) = |N_A(v)|$ for the *degree of* v *into* A,

$e(A) = e(G[A])$ for the number of edges (of G) inside A and

$e(A, B)$ for the number of edges ab of G with $a \in A$ and $b \in B$.

Warnings!

Some authors write P_n for the path with n edges, not n vertices.

In some books 'graph' is used to mean 'multi-graph' – a variant where multiple edges between two vertices are allowed, and maybe edges from a vertex to itself. In most such books a 'simple graph' is what we call a graph.

Some people write $G \setminus e$ (*NOT* G/e) for $G - e$, and $G \setminus v$ for $G - v$.

You may also see $v(G)$ or $n(G)$ instead of $|G|$.

The term *size* is used in different ways by different people. Best to avoid and stick with $e(G)$ or 'number of edges'.

If you find an error please check the website, and if it has not already been corrected, e-mail: `Paul.Balister@maths.ox.ac.uk`.