B8.5 Graph Theory

Sheet 0 — MT20

Not for classes

The following problems (mostly mentioned in the lectures/notes) are primarily intended for students who did not do Part A Graph Theory, to help you get yourself up to speed. Other students may find them useful too. They will not be discussed in classes; solutions will be posted on the course website.

You don't need to do all the questions, but it may be helpful. I suggest comparing your answers to the model solutions one (or a few) at a time; having seen the model solution, try to write the next answer in a similar style/level of detail.

1. Let x and y be vertices of a graph G. Show that G contains an (i.e., at least one) x-y walk if and only if G contains an x-y path.

2. Let $G = (V, E)$ be a graph, and define a relation $\sim$ on V by $x \sim y$ if x and y are connected in G, i.e., if there is an x-y path/walk in G (possibly of length 0). Show, giving full details, that $\sim$ is an equivalence relation.

3. [A little tedious; omit if you like.] Check that any graph G is the disjoint union of its components (maximal connected subgraphs). It may help to first show that the components correspond to equivalence classes of the relation $\sim$ in the previous question.

4. Show that TFAE (The Following Are Equivalent): (a) T is a tree, (b) T is a minimal (w.r.t. edges) connected graph, (c) T is a maximal (w.r.t. edges) acyclic graph.

5. Modify the argument in lectures to show that any tree with at least two vertices has at least two leaves.

6. Let T be a tree with $|T| \geq 2$, and let P be a longest path in T. Prove, giving full details, that the ends of P are leaves. Deduce that T has at least two leaves.

7. Show that any two vertices of a tree T are joined by a *unique* path in T.

8. Let $(d_1, \ldots, d_n)$ be a sequence of integers with $n \geq 2$. Show that there is a tree on $[n]$ with $d(i) = d_i$ for each i if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^{n} d_i = 2n - 2$.

9. Show that deleting any edge from a tree T leaves a graph with exactly two components. Show that deleting a vertex v leaves $d(v)$ components. [Hint: you could do this directly, or try a short cut using what we know about numbers of edges in trees.]

Paul Balister: Paul.Balister@maths.ox.ac.uk