

B8.5 Graph Theory

Sheet 1 — MT20

1. Prove that if a graph G has at least two vertices then G contains two vertices of the same degree.
2. (a) Find all connected graphs G that do not contain the complete bipartite graph $K_{1,2}$ as a subgraph.
(b) Find all connected graphs G that do not contain $K_{1,2}$ as an *induced* subgraph.
3. Find all connected graphs G with $\Delta(G) \leq 2$.
4. A graph is r -regular if every vertex has degree exactly r . Prove that there is a 3-regular graph on n vertices if and only if n is even and $n \geq 4$.
5. Show that if T is a tree that is not a path, then T has at least three leaves. Can you classify all trees with exactly three leaves?
6. Let $f(n)$ be the number of isomorphism classes of connected n -vertex graphs in which every vertex has degree at most 3. Show that $f(n) \rightarrow \infty$. Show further that there are constants $A, c > 0$ such that $f(n) \geq Ae^{cn}$ for every n .
7. Find (draw) the tree on $[9]$ with Prüfer code 6423743.
8. Consider the algorithm in the lecture notes mapping a Prüfer code $\mathbf{c} = (c_1, c_2, \dots, c_{n-2})$ to a tree T on $[n]$. Show that T has code \mathbf{c} .
9. Let $k \geq 1$, and suppose that G is a connected $2k$ -regular graph.
 - (a) Prove that if G has an even number of edges then there is a k -regular subgraph H of G such that $V(H) = V(G)$. [Hint: G has an Euler circuit.]
 - (b) What can you say if G has an odd number of edges?
10. The *discrete cube* Q_n has vertex set $\{0, 1\}^n$, and two vertices are joined if they differ in exactly one coordinate. (Thus $|Q_n| = 2^n$ and $e(Q_n) = n2^{n-1}$.) Prove that Q_n contains a Hamilton cycle for every $n \geq 2$.
11. For each integer $k \geq 1$, find a connected, *non-complete* graph G containing no P_{2k+2} (path of edge length $2k + 1$) with $\bar{d}(G) \geq 2k - 0.0001$. (Hint: try $k = 1$ first.)

Optional bonus questions. These may not be covered in classes; MFoCS students should attempt them.

12. For $1 \leq k \leq n$ a graph G on $[n]$ is a $[k]$ -forest if it is acyclic and has exactly k components, with the vertices $1, 2, \dots, k$ in distinct components. Let $a_{n,k}$ be the number of $[k]$ -forests on $[n]$, and set $a_{n,0} = 0$. Show that

$$a_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} a_{n-1,k-1+i}$$

for any $n \geq 2$ and $1 \leq k \leq n$, and hence that $a_{n,k} = kn^{n-k-1}$. Deduce Cayley's formula. [Hint: recall the Binomial Theorem, and also that $\binom{a}{b} = \frac{a}{b} \binom{a-1}{b-1}$ for $a \geq b \geq 1$.]

13. The *average degree* of $G = (V, E)$ is $|V|^{-1} \sum_{v \in V} d(v)$. Let $G = (V, E)$ be a graph with average degree d and without isolated vertices.
- Show that there is a vertex $v \in V$ so that the average degree of the neighbours of v is at least d .
 - Must there be a vertex $v \in V$ so that the average degree of the neighbours of v is at most d ?
14. Suppose that G and H are infinite graphs, and that G is isomorphic to a subgraph of H and H is isomorphic to a subgraph of G . Must G and H be isomorphic?