

# B8.5 Graph Theory

## Sheet 1 — MT20

1. Prove that if a graph  $G$  has at least two vertices then  $G$  contains two vertices of the same degree.
2. (a) Find all connected graphs  $G$  that do not contain the complete bipartite graph  $K_{1,2}$  as a subgraph.  
(b) Find all connected graphs  $G$  that do not contain  $K_{1,2}$  as an *induced* subgraph.
3. Find all connected graphs  $G$  with  $\Delta(G) \leq 2$ .
4. A graph is  $r$ -*regular* if every vertex has degree exactly  $r$ . Prove that there is a 3-regular graph on  $n$  vertices if and only if  $n$  is even and  $n \geq 4$ .
5. Show that if  $T$  is a tree that is not a path, then  $T$  has at least three leaves. Can you classify all trees with exactly three leaves?
6. Let  $f(n)$  be the number of isomorphism classes of connected  $n$ -vertex graphs in which every vertex has degree at most 3. Show that  $f(n) \rightarrow \infty$ . Show further that there are constants  $A, c > 0$  such that  $f(n) \geq Ae^{cn}$  for every  $n$ .
7. Find (draw) the tree on  $[9]$  with Prüfer code 6423743.
8. Consider the algorithm in the lecture notes mapping a Prüfer code  $\mathbf{c} = (c_1, c_2, \dots, c_{n-2})$  to a tree  $T$  on  $[n]$ . Show that  $T$  has code  $\mathbf{c}$ .
9. Let  $k \geq 1$ , and suppose that  $G$  is a connected  $2k$ -regular graph.
  - (a) Prove that if  $G$  has an even number of edges then there is a  $k$ -regular subgraph  $H$  of  $G$  such that  $V(H) = V(G)$ . [Hint:  $G$  has an Euler circuit.]
  - (b) What can you say if  $G$  has an odd number of edges?
10. The *discrete cube*  $Q_n$  has vertex set  $\{0, 1\}^n$ , and two vertices are joined if they differ in exactly one coordinate. (Thus  $|Q_n| = 2^n$  and  $e(Q_n) = n2^{n-1}$ .) Prove that  $Q_n$  contains a Hamilton cycle for every  $n \geq 2$ .
11. For each integer  $k \geq 1$ , find a connected, *non-complete* graph  $G$  containing no  $P_{2k+2}$  (path of edge length  $2k + 1$ ) with  $\bar{d}(G) \geq 2k - 0.0001$ . (Hint: try  $k = 1$  first.)

*Optional bonus questions. These may not be covered in classes; MFoCS students should attempt them.*

12. For  $1 \leq k \leq n$  a graph  $G$  on  $[n]$  is a  $[k]$ -forest if it is acyclic and has exactly  $k$  components, with the vertices  $1, 2, \dots, k$  in distinct components. Let  $a_{n,k}$  be the number of  $[k]$ -forests on  $[n]$ , and set  $a_{n,0} = 0$ . Show that

$$a_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} a_{n-1,k-1+i}$$

for any  $n \geq 2$  and  $1 \leq k \leq n$ , and hence that  $a_{n,k} = kn^{n-k-1}$ . Deduce Cayley's formula. [Hint: recall the Binomial Theorem, and also that  $\binom{a}{b} = \frac{a}{b} \binom{a-1}{b-1}$  for  $a \geq b \geq 1$ .]

13. The *average degree* of  $G = (V, E)$  is  $|V|^{-1} \sum_{v \in V} d(v)$ . Let  $G = (V, E)$  be a graph with average degree  $d$  and without isolated vertices.
- (a) Show that there is a vertex  $v \in V$  so that the average degree of the neighbours of  $v$  is at least  $d$ .
  - (b) Must there be a vertex  $v \in V$  so that the average degree of the neighbours of  $v$  is at most  $d$ ?
14. Suppose that  $G$  and  $H$  are infinite graphs, and that  $G$  is isomorphic to a subgraph of  $H$  and  $H$  is isomorphic to a subgraph of  $G$ . Must  $G$  and  $H$  be isomorphic?