B8.5 Graph Theory

Sheet 4 — MT20

1. (a) Is there a network $(\overrightarrow{G}, s, t, c)$ containing edges xy and yx such that there are two maximal *acyclic* flows f_1 and f_2 with $f_1(xy) > 0$ and $f_2(yx) > 0$? (Recall a flow is acyclic if there is no directed cycle $x_0x_1 \cdots x_n = x_0$ with $f(x_ix_{i+1}) > 0$ for all i.)

   (b) What if the flows are 'efficient', i.e., minimise $\sum_{xy} |f(xy)|$?

2. Prove the Fan Lemma: Let $n \geq k+1$. An n-vertex graph G is k-connected if and only if for every choice $U \subseteq V(G)$ with $|U| = k$ and $x \in V(G) \setminus U$, G has k paths from x to U that are vertex disjoint except at x.

3. Prove that if G is k-connected, where $k \geq 2$, then every set of k vertices of G lies on a cycle. Is the converse true?

4. Let G be a bipartite graph with bipartition (V_1, V_2).

   (a) For $0 \leq d \leq |V_1|$, show that G contains a matching of size $|V_1| - d$ if and only if $|N(S)| \geq |S| - d$ for every $S \subseteq V_1$.

   (b) Show that G contains a 1-to-r matching (i.e., a 'pairing' of each $v \in V_1$ with a set $A_v \subseteq N(v)$ such that $|A_v| = r$ for all $v \in V_1$ and the sets A_v are disjoint) if and only if $|N(S)| \geq r|S|$ for every $S \subseteq V_1$.

5. For each $1 \leq k \leq \ell \leq d$, construct a graph G with $\kappa(G) = k$, $\lambda(G) = \ell$, and $\delta(G) = d$.

6. Show that every 3-regular graph either contains a 1-factor, or contains at least 3 bridges.

7. Determine $\mathrm{ex}(n, K_{1,3})$ for every n. Describe the extremal graphs.

8. Determine $\mathrm{ex}(n, 2K_2)$ for every n. What are the extremal graphs? (Here $2K_2$ means a pair of vertex-disjoint edges.)

9. Suppose that G is a graph with $n > r+1$ vertices and $t_r(n) + 1$ edges.

   (a) Prove that for every p with $r+1 < p \leq n$ there is a subgraph H of G with $|H| = p$ and $e(H) \geq t_r(p) + 1$.

   (b) Prove that G contains two copies of K_{r+1} with exactly r common vertices.

10. *For those who did not do Part A Graph Theory:* We say that an n-by-n matrix is *doubly stochastic* if all its entries are nonnegative and every row and every column sums to 1. A matrix is a *permutation matrix* if it is doubly stochastic and all entries are 0 or 1 (i.e., every row and column contains a single 1, and all other entries are 0).

    Prove that every doubly stochastic matrix is a convex combination of permutation matrices.

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Optional bonus questions. These may not be covered in classes; MFoCS students should attempt them.

11. Suppose G is a connected graph with minimum degree $\delta(G) = k$. Show that there is a path $x_1 \cdots x_k$ such that $G - \{x_1, \dots, x_k\}$ is connected. [Hint: consider a longest path $x_1 \cdots x_\ell$ in G. If $G - \{x_1, \dots, x_k\}$ is not connected, consider the neighbours of an endvertex of a longest path in a component C of $G - \{x_1, \dots, x_k\}$, $x_{k+1} \notin C$, that ends at a vertex adjacent to one of the x_i, $i \leq k$.]

12. Let H be a graph, and define $c_n(H) := \mathrm{ex}(n, H)/\binom{n}{2}$. Prove that $c_n(H) \leq c_{n-1}(H)$, and show that $\lim_{n\to\infty} c_n(H)$ exists.

13. Let $k \geq 2$ and let G be a graph with $|G| = n \geq 2k - 1$. Prove that if $e(G) \geq (2k-3)(n-k+1)+1$ then G contains a subgraph H such that H is k-connected.

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