Dynamics: Problem Sheet 6 (of 8)

1. Suppose that a particle moves in response to a central force per unit mass $-f(r)\,\mathbf{e}_r$, where

$$f(r) \quad = \quad \frac{\alpha}{r^2} + \frac{\beta^2}{r^3} \; .$$

Here r denotes distance to the origin and α, β are constants. Initially the particle is at $r = \beta^2/3\alpha$, $\theta = 0$ and is moving with speed $4\alpha/\beta$ in a direction making an angle of $\pi/3$ with the radius vector pointing towards the origin.

Starting from Newton's second law show that, if $u = 1/r$, then

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \frac{u}{4} \quad = \quad \frac{3\alpha}{4\beta^2} \; ,$$

with

$$u \quad = \quad \frac{3\alpha}{\beta^2} \; , \qquad \frac{\mathrm{d}u}{\mathrm{d}\theta} \quad = \quad \frac{\alpha\sqrt{3}}{\beta^2} \qquad \text{when } \theta \; = \; 0 \; .$$

Hence show that the solution is

$$\frac{1}{r} \quad = \quad \frac{3\alpha}{\beta^2}\left(\frac{2}{\sqrt{3}}\sin\frac{\theta}{2} + 1\right) \; .$$

Sketch the orbit.

2. A particle is dropped from the top of a tower on the Earth's equator. As a result of the Earth's rotation, does it land slightly to the East, or slightly to the West of the tower?

3. A particle of mass m is acted on by a central force $-F(r)\mathbf{e}_r$.

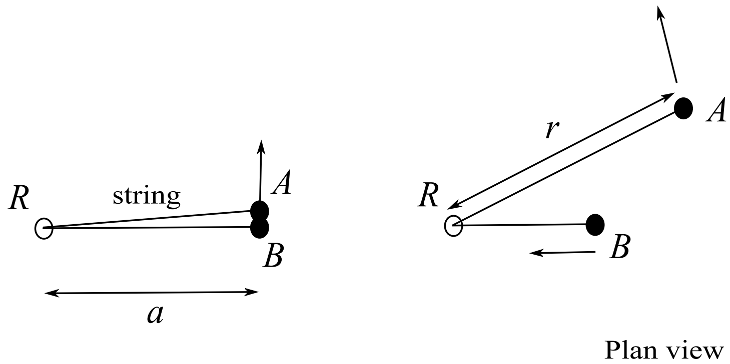
   (a) Given the angular momentum per unit mass of the particle, $h = r^2\dot{\theta}$, show that a circular orbit is possible providing

$$F(a) = \frac{mh^2}{a^3}.$$

   (b) With h fixed show that this circular orbit is stable providing

$$3F(a) + aF'(a) > 0.$$

4. Two particles A, B of mass m_1, m_2, respectively, lie together on a smooth horizontal table.



They are connected by a light inextensible string of length $2a$ which passes through a light ring R fixed in the table at a distance a from the particles. The ring is smooth and can rotate freely. The particle A is given an initial velocity perpendicular to the string in the plane of the table.

Show that if $u = 1/r$, where r is the distance of A from R, then

$$\frac{d^2u}{d\theta^2} + \frac{m_1}{m_1+m_2}u \quad = \quad 0 \ ,$$

where θ is the angle ARB. Hence find the equation of the path taken by A (up until the moment B reaches R).

[*Hint*: The tension in the string provides a central force for both particles.]

Please send comments and corrections to `gaffney@maths.ox.ac.uk`.