Numerical Solution of Differential Equations I: Problem Sheet 0

What is this course about?

[This is an illustrative sheet, which will not be discussed in the problem classes.]

This course is concerned with the construction and mathematical analysis of numerical methods for the approximate solution of differential equations. The MT part of the course (NSDE I) focuses on numerical methods for initial-value problems for ordinary differential equations, and initial-value problems and initial-boundary-value problems for parabolic equations typified by the unsteady heat equation.

This illustrative sheet will not be discussed in the classes as it also includes model answers. Its purpose is simply to provide a flavor of the lecture course.

1. Euler's method for the numerical solution of an initial-value problem for an ordinary differential equation of the form $y' = f(x, y)$, $y(x_0) = y_0$, where f is a given function and x_0 and y_0 are given real numbers, aims to compute an approximate solution of this initial-value problem on a given closed interval $[x_0, X_M]$ of the real line, where $X_M > x_0$, by subdividing the interval $[x_0, X_M]$ into a number of subintervals of length $h > 0$ each, where $h = (X_M - x_0)/N$, and N is (typically) a (large) positive integer.

   The points of subdivision in the interval $[x_0, X_M]$ are denoted by x_n, where $x_n = x_0 + nh$, $n = 0, 1, \ldots, N$, and approximations to the values $y(x_n)$ of the unknown exact solution y of the initial-value problem at the points x_n, $n = 1, 2, \ldots, N$, are computed by using the formula (called *Euler's method*):

$$\frac{y_n - y_{n-1}}{h} = f(x_{n-1}, y_{n-1}), \qquad n = 1, \ldots, N, \quad \text{with } y_0 \text{ given.}$$

   For example, if we are to calculate an approximate solution of the initial-value problem

$$y' = x^2 + y^2, \qquad y(0) = 0$$

   on the interval $x \in [0, 1]$, Euler's method will compute a sequence of approximations y_n defined by

$$\frac{y_n - y_{n-1}}{h} = x_{n-1}^2 + y_{n-1}^2, \qquad n = 1, \ldots, N, \quad \text{with } y_0 = 0,$$

   where $x_n = x_0 + nh = 0 + nh = nh$, for $n = 0, 1 \ldots, N$. By choosing $N = 10$, say, we have $h = (1 - 0)/N = 1/10$, and thereby we obtain:

$$y_1 = y_0 + h(x_0^2 + y_0^2) = 0 + \frac{1}{10}(0^2 + 0^2) = 0,$$
$$y_2 = y_1 + h(x_1^2 + y_1^2) = 0 + \frac{1}{10}((1/10)^2 + 0^2) = 0.0010.$$

   I leave it to you to verify that $y_3 = 0.0050$, $y_4 = 0.0140$, and, if you are in the mood, you can also compute $y_5, \ldots, y_{10}$ by hand, but as you can see this is a rather tedious exercise. So let us make our lives simpler!

   Here is a small MATLAB code that will simplify this task, and which will, for any choice of $N > 1$, compute the desired sequence of Euler approximations. For example, in the case of $N = 10$ (you could try to change N), the code is this:

```
clear, set(0,'DefaultFigureWindowStyle','docked')
N = 10; h = 1/N;
Eul = nan(1, N+1);
y0 = 0; Eul(1) = y0;

for ii = 1:N
    Eul(ii+1) = Eul(ii) + h*(((ii-1)*h)^2 + (Eul(ii))^2);
end

t = linspace(0, 1, N+1);figure(1);
plot(t, Eul, '*-')
legend({'Eul'},'location','northwestoutside','FontSize',24)
```

Using this code, you can easily compute (rounded to four decimal places)

$$y_5 = 0.0300, \quad y_6 = 0.0551, \quad y_7 = 0.0914, \quad y_8 = 0.1413, \quad y_9 = 0.2072, \quad y_{10} = 0.2925,$$

the numerical approximations to the unknown values $y(0.5)$, $y(0.6)$, $y(0.7)$, $y(0.8)$, $y(0.9)$ and $y(1)$ of the exact solution y at $x = 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

2. Change N to a different number in the above MATLAB code.

   What do you observe as N increases (say $N = 2$, $N = 5$, $N = 10$, $N = 20$, $N = 50$, $N = 100$)?

   Here are some questions you might like to ponder (but I wouldn't suggest that you try to answer them, as that is precisely our objective in this lecture course).

   We would like to prove that as $N \to \infty$ the sequence of approximations produced by Euler's method approaches the unknown exact solution of the initial-value problem. How can we do this, when the exact solution is not known?

   How quickly does the sequence of approximations computed by using Euler's method approach the unknown exact solution of the initial-value problem as $N \to \infty$?

   Are there numerical methods which produce approximations that approach the exact solution even faster than the ones computed by means of Euler's method?

   How sensitive is the computed solution to perturbations of the initial datum, y_0? How can we ensure that small changes in y_0 give rise to small changes in the subsequent numerical approximations y_1, y_2, ...?

   Suppose now that the equation whose unknown exact solution is to be approximated is not an ordinary differential equation, but a partial differential equation. How do we construct a numerical method for its approximate solution, and what mathematical tools are there to explore the accuracy of the resulting numerical approximations?

   These are only some of the many interesting and relevant mathematical questions we shall try to answer in this lecture course.

   As ordinary and partial differential equations arise is numerous applications in the physical sciences, engineering, life sciences, environmental sciences, economics, financial modelling, meteorology and climatology, to name just a few, it is important to create a solid body of mathematical theory that can answer questions of this kind.