Numerical Solution of Differential Equations I

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Lecture 1

Picard's Theorem

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The ODE (1) will be considered together with an **initial condition**: given two real numbers x_0 and y_0, find a solution to (1) for $x > x_0$ such that

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The problem (1), (2) is called an **initial-value problem**.

Theorem (Picard's Theorem[1])

Suppose that $f(\cdot,\cdot)$ is a continuous function of its arguments in a region U of the (x, y) plane which contains the rectangle

$$\mathrm{R} := \{(x, y) \,:\, x_0 \le x \le X_M, \quad |y - y_0| \le Y_M\},$$

where $X_M > x_0$ and $Y_M > 0$ are constants.

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where $X_M > x_0$ and $Y_M > 0$ are constants.
Suppose also, that there exists a positive constant L such that

$$|f(x, y) - f(x, z)| \le L|y - z| \tag{3}$$

holds whenever (x, y) and (x, z) lie in the rectangle R.

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$$M := \max\{|f(x, y)| \,:\, (x, y) \in \mathrm{R}\},$$

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$$M := \max\{|f(x,y)| \,:\, (x,y) \in \mathrm{R}\},$$

suppose that $M(X_M - x_0) \le Y_M$. Then, there exists a unique continuously differentiable function $x \mapsto y(x)$, defined on the closed interval $[x_0, X_M]$, which satisfies (1) and (2).

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\begin{aligned}
& y_0(x) \equiv y_0, \\
& y_n(x) := y_0 + \int_{x_0}^{x} f(\xi, y_{n-1}(\xi))\,\mathrm{d}\xi, \qquad n = 1, 2, \ldots,
\end{aligned} \tag{4}
$$

and show, using the conditions of the theorem, that $\{y_n\}_{n=0}^{\infty}$, as a sequence of continuous functions, converges uniformly on the interval $[x_0, X_M]$ to a continuous function y defined on $[x_0, X_M]$ s.t.

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y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))\,\mathrm{d}\xi.
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This implies that y is continuously differentiable on $[x_0, X_M]$ and it satisfies the differential equation (1) and the initial condition (2). The uniqueness of the solution follows from the Lipschitz condition.

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Picard's Theorem has a natural extension to an initial-value problem for a system of m differential equations of the form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(x_0) = \mathbf{y}_0, \tag{5}$$

where $\mathbf{y}_0 \in \mathbb{R}^m$ and $\mathbf{f} : [x_0, X_M] \times \mathbb{R}^m \to \mathbb{R}^m$.

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Consider the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^m$ defined by

$$\|v\| := \left(\sum_{i=1}^{m} |v_i|^2 \right)^{1/2}, \qquad v \in \mathbb{R}^m.$$

We then have the following result.

Theorem (Picard's Theorem for systems)

Suppose that $\mathbf{f}(\cdot,\cdot)$ *is a continuous function of its arguments in a region* U *of the* $(x,\mathbf{y})$ *space* $\mathbb{R}^{1+m}$ *containing the parallelepiped*

$$\mathrm{R} := \{(x,\mathbf{y}) : x_0 \le x \le X_M, \quad \|\mathbf{y}-\mathbf{y}_0\| \le Y_M\},$$

where $X_M > x_0$ *and* $Y_M > 0$ *are constants.*

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where $X_M > x_0$ *and* $Y_M > 0$ *are constants.*
Suppose also that there exists a positive constant L *such that*

$$\|\mathbf{f}(x,\mathbf{y}) - \mathbf{f}(x,\mathbf{z})\| \le L\|\mathbf{y}-\mathbf{z}\| \tag{6}$$

holds whenever $(x,\mathbf{y})$ *and* $(x,\mathbf{z})$ *lie in* R. *Finally, letting*

$$M := \max\{\|\mathbf{f}(x,\mathbf{y})\| \,:\, (x,\mathbf{y}) \in \mathrm{R}\},$$

suppose that $M(X_M - x_0) \le Y_M$. *Then, there exists a unique continuously differentiable function* $x \mapsto \mathbf{y}(x)$, *defined on the closed interval* $[x_0, X_M]$, *which satisfies (5).*

We conclude by introducing the notion of *stability*.

Definition

A solution $\mathbf{y} = \mathbf{y}(x)$ to (5) is said to be **stable** on the interval $[x_0, X_M]$ if:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \text{s.t.} \quad \forall\, \mathbf{z}_0 \in \mathbb{R}^m \quad \text{satisfying} \quad \|\mathbf{y}_0 - \mathbf{z}_0\| < \delta$$

the solution $\mathbf{z} = \mathbf{z}(x)$ to the differential equation $\mathbf{z}' = \mathbf{f}(x, \mathbf{z})$ satisfying the initial condition $\mathbf{z}(x_0) = \mathbf{z}_0$ is defined for all $x \in [x_0, X_M]$ and satisfies

$$\|\mathbf{y}(x) - \mathbf{z}(x)\| < \varepsilon \qquad \text{for all } x \text{ in } [x_0, X_M].$$

Using this definition, we can state the following theorem.

Theorem

Under the hypotheses of Picard's Theorem, the (unique) solution $\mathbf{y} = \mathbf{y}(x)$ to the initial-value problem (5) is stable on the interval $[x_0, X_M]$, (where we assume that $-\infty < x_0 < X_M < \infty$).

PROOF: Since

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(\xi, \mathbf{y}(\xi))\,\mathrm{d}\xi$$

and

$$\mathbf{z}(x) = \mathbf{z}_0 + \int_{x_0}^{x} \mathbf{f}(\xi, \mathbf{z}(\xi))\,\mathrm{d}\xi,$$

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it follows that

$$\begin{aligned} \|\mathbf{y}(x) - \mathbf{z}(x)\| &\leq \|\mathbf{y}_0 - \mathbf{z}_0\| + \int_{x_0}^{x} \|\mathbf{f}(\xi, \mathbf{y}(\xi)) - \mathbf{f}(\xi, \mathbf{z}(\xi))\|\,\mathrm{d}\xi \\ &\leq \|\mathbf{y}_0 - \mathbf{z}_0\| + L\int_{x_0}^{x} \|\mathbf{y}(\xi) - \mathbf{z}(\xi)\|\,\mathrm{d}\xi. \end{aligned} \tag{7}$$

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Now put $A(x) := \|\mathbf{y}(x) - \mathbf{z}(x)\|$ and $a := \|\mathbf{y}_0 - \mathbf{z}_0\|$; then, (7) can be written as

$$A(x) \leq a + L\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi, \qquad x_0 \leq x \leq X_M. \qquad (8)$$

Multiplying (8) by $\exp(-Lx)$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-Lx}\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi\right] \leq a\mathrm{e}^{-Lx}. \tag{9}$$

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Integrating the inequality (9), we deduce that

$$\mathrm{e}^{-Lx}\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi \le \frac{a}{L}\left(\mathrm{e}^{-Lx_0} - \mathrm{e}^{-Lx}\right),$$

that is

$$L\int_{x_0}^{x} A(\xi)\,\mathrm{d}\xi \le a\left(\mathrm{e}^{L(x-x_0)} - 1\right). \tag{10}$$

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Now substituting (10) into (8) gives

$$A(x) \leq a\,\mathrm{e}^{L(x-x_0)}, \qquad x_0 \leq x \leq X_M. \tag{11}$$

[The implication "(8) $\Rightarrow$ (11)" is called **Gronwall's Lemma**.]

Returning to our original notation, we deduce from (11) that

$$\|\mathbf{y}(x) - \mathbf{z}(x)\| \le \|\mathbf{y}_0 - \mathbf{z}_0\| \, \mathrm{e}^{L(x-x_0)}, \qquad x_0 \le x \le X_M. \tag{12}$$

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Thus, given $\varepsilon > 0$ as in Definition 3, we choose

$$\delta = \varepsilon \, \mathrm{e}^{-L(X_M - x_0)}.$$

Then, if $\|\mathbf{y}_0 - \mathbf{z}_0\| < \delta$, it follows from (12) that

$$\|\mathbf{y}(x) - \mathbf{z}(x)\| < \varepsilon \, \mathrm{e}^{-L(X_M - x_0)} \, \mathrm{e}^{L(x-x_0)} \leq \varepsilon, \qquad x_0 \leq x \leq X_M.$$

Therefore, the solution $\mathbf{y}$ is stable, as has been asserted. $\diamond$

Remark

A solution which is stable on $[x_0, \infty)$ *(i.e. stable on* $[x_0, X_M]$ *for each* X_M *and with* δ *independent of* X_M*) is said to be* **stable in the sense of Lyapunov**.

Moreover, if

$$\lim_{x \to \infty} \|\mathbf{y}(x) - \mathbf{z}(x)\| = 0,$$

then the solution $\mathbf{y} = \mathbf{y}(x)$ *is called* **asymptotically stable**.