Numerical Solution of Differential Equations I

Endre Süli

Mathematical Institute
University of Oxford
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Lecture 5

 

Carl David Tolmé Runge (30 August 1856 – 3 January 1927)
Martin Wilhelm Kutta (3 November 1867 – 25 December 1944)

Absolute stability of Runge–Kutta methods

It is instructive to consider the model problem

$$y' = \lambda y, \quad y(0) = y_0 \ (\neq 0), \tag{1}$$

with $\lambda \in \mathbb{R}_{<0}$. The analytical solution to this initial value problem,

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For simplicity we restrict ourselves to the case of R-stage methods of order of accuracy R, with $1 \leq R \leq 4$.

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For such h the explicit Euler method is said to be **absolutely stable** and the interval $(-2, 0)$ is referred to as the **interval of absolute stability** of the method.

$R = 2$

This corresponds to two-stage second-order Runge–Kutta methods:

$$y_{n+1} = y_n + h(c_1 k_1 + c_2 k_2),$$

where

$$k_1 = f(x_n, y_n), \qquad k_2 = f(x_n + a_2 h, y_n + b_{21} h k_1)$$

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Hence the method is absolutely stable if, and only if,

$$\left|1 + \bar{h} + \frac{1}{2}\bar{h}^2\right| < 1, \qquad \text{i.e. when } \bar{h} \in (-2, 0).$$

$R = 3$

An analogous argument shows that

$$y_{n+1} = \left(1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3\right) y_n.$$

Demanding that

$$\left|1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3\right| < 1$$

then yields the interval of absolute stability: $\bar{h} \in (-2.51, 0)$.

$R = 4$

We have that

$$y_{n+1} = \left(1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3 + \frac{1}{24}\bar{h}^4\right) y_n,$$

and the associated interval of absolute stability is $\bar{h} \in (-2.78, 0)$.

$R \geq 5$

By applying the Runge–Kutta method to the model problem (1) still results in a recursion of the form

$$y_{n+1} = A_R(\bar{h})y_n, \qquad n \geq 0.$$

However, unlike the case when $R = 1, 2, 3, 4$, in addition to $\bar{h}$ now $A_R(\bar{h})$ also depends on the coefficients of the Runge–Kutta method.

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Remark. The analysis above can be extended to the case when $\lambda \in \mathbb{C}$ and $\mathrm{Re}(\lambda) < 0$.

Regions of absolute stability of RK methods plotted in the complex plane

Consider $y' = \lambda y$, $y(0) = y_0 (\neq 0)$, with $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) < 0$.

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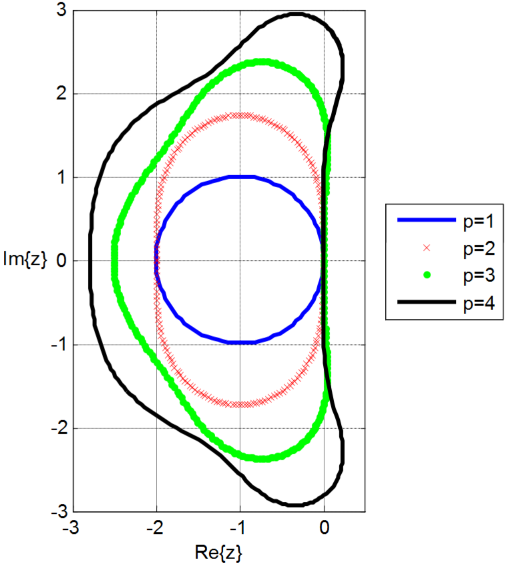

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Example

RK4 involves 4 function evaluations per step. For comparison, by considering three consecutive points x_{n-1}, $x_n = x_{n-1} + h$, $x_{n+1} = x_{n-1} + 2h$, integrating the ODE between x_{n-1} and x_{n+1},

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\begin{aligned}
y(x_{n+1}) &= y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x, y(x))\,\mathrm{d}x \\
&\approx y(x_{n-1}) + \frac{1}{3}h\left[f(x_{n-1}, y(x_{n-1})) + 4f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right]
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thanks to Simpson's rule. This leads to the method

$$y_{n+1} = y_{n-1} + \frac{1}{3}h\left[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right].$$

In contrast with one-step methods, where only a single value y_n was needed to compute the next approximation y_{n+1}, here we need *two* preceding values, y_n and y_{n-1} to be able to calculate y_{n+1}, and therefore the method in the last example is not a one-step method.

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This is an example of a **linear multi-step method**.

Given a sequence of equally spaced mesh points (x_n) with step size h, we consider the general **linear k-step method**

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}), \tag{2}$$

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The numerical method (2) is called *linear* because it involves only linear combinations of the $\{y_n\}$ and the $\{f(x_n, y_n)\}$; for simplicity, we shall write f_n instead of $f(x_n, y_n)$.

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c) The four-step **Adams–Bashforth method**

$$y_{n+4} = y_{n+3} + \frac{1}{24}h[55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n]$$

is an explicit linear four-step method.