Numerical Solution of Differential Equations I

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Lecture 6

Analysis of linear multi-step methods

Consider the general **linear k-step method**

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}),$$

where the coefficients $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_k$ are real constants, $\alpha_k \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$.

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We introduce the concepts of

- stability,
- consistency, and
- convergence.

Zero-stability

We need k starting values, $y_0, \ldots, y_{k-1}$, before we can apply a linear k-step method to an initial-value problem. Of these, y_0 is given by the initial condition $y(x_0) = y_0$, but $y_1, \ldots, y_{k-1}$, have to be computed by other means (e.g. by using a Runge–Kutta method). Thus, the starting values will contain numerical errors.

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Question: How do these errors affect further approximations y_n, $n \geq k$, which are calculated by means of a k-step method.

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Question: How do these errors affect further approximations y_n, $n \geq k$, which are calculated by means of a k-step method.

We consider the 'stability' of the numerical method with respect to 'small perturbations' in the starting conditions.

Definition

A linear k-step method for the ordinary differential equation $y' = f(x, y)$ is said to be **zero-stable** if there exists a constant K such that, for any two sequences (y_n) and $(\hat{y}_n)$, which have been generated by the same formulae but with different initial data $y_0,\ y_1, \ldots, y_{k-1}$ and $\hat{y}_0,\ \hat{y}_1, \ldots, \hat{y}_{k-1}$, respectively, we have

$$|y_n - \hat{y}_n| \leq K \max\{|y_0 - \hat{y}_0|, |y_1 - \hat{y}_1|, \ldots, |y_{k-1} - \hat{y}_{k-1}|\}$$

for $x_n \leq X_M$, and as h tends to 0.

Remark

We shall prove that zero-stability of a linear multistep method can be checked by considering its behaviour when applied to the trivial differential equation $y' = 0$; it is for this reason that the kind of stability expressed in Definition 1 is called *zero* stability.

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While Definition 1 is expressive in the sense that it conforms with the intuitive notion of stability whereby "small perturbations at input give rise to small perturbations at output", it would be tedious to verify the zero-stability of a linear multi-step method using Definition 1; thus we shall formulate an algebraic equivalent of zero-stability, known as the **root condition**, to simplify this task.

Given the linear k-step method we consider its **first** and **second characteristic polynomial**, respectively

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j,$$
$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j,$$

where, as before, we assume that

$$\alpha_k \neq 0, \quad \alpha_0^2 + \beta_0^2 \neq 0.$$

Theorem (Root condition)

A linear multi-step method is zero-stable for any ordinary differential equation $y' = f(x, y)$, where f satisfies the Lipschitz condition, if, and only if, its first characteristic polynomial has zeros inside the closed unit disc, with any which lie on the unit circle being simple.

Proof:

Necessity. Consider the linear k-step method, applied to $y' = 0$:

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_1 y_{n+1} + \alpha_0 y_n = 0. \qquad (1)$$

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The general solution of this kth order linear difference equation is[1]

$$y_n = \sum_s p_s(n) z_s^n,$$

where z_s is a zero of the first characteristic polynomial $\rho(z)$ and the polynomial $p_s(\cdot)$ has degree one less than the multiplicity of the zero.

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Clearly, if $|z_s| > 1$ then there are starting values for which the corresponding solutions grow like $|z_s|^n$ and if $|z_s| = 1$ and its multiplicity is $m_s > 1$ then there are solutions growing like $n^{m_s - 1}$. In either case there are solutions that grow unbounded as $n \to \infty$, i.e. as $h \to 0$ with nh fixed.

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Considering starting data $y_0, y_1, \ldots, y_{k-1}$ which give rise to such an unbounded solution (y_n), and starting data

$$\hat{y}_0 = \hat{y}_1 = \cdots = \hat{y}_{k-1} = 0$$

for which the corresponding solution of (1) is $(\hat{y}_n)$ with $\hat{y}_n = 0$ for all n, we see that the inequality in the definition of zero stability cannot hold.

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To summarise, if the root condition is violated then the method is not zero-stable.

Sufficiency. The proof that the root condition is sufficient for zero-stability is long and technical, and will be omitted here. For details, see, for example, P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, New York, 1962. $\diamond$

Example

a) The explicit and implicit Euler methods have first characteristic polynomial $\rho(z) = z - 1$ with simple root $z = 1$, so both methods are zero-stable.
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The same is true of the trapezium method.

b) The Adams–Bashforth method considered in an earlier example has the first characteristic polynomial $\rho(z) = z^3(z-1)$ and is therefore zero-stable.

c) The three-step (sixth order consistent) linear multi-step method

$$11y_{n+3} + 27y_{n+2} - 27y_{n+1} - 11y_n = 3h[f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n]$$

is *not* zero-stable. Indeed, the associated first characteristic polynomial $\rho(z) = 11z^3 + 27z^2 - 27z - 11$ has roots at $z_1 = 1$, $z_2 \approx -0.3189$, $z_3 \approx -3.1356$, so $|z_3| > 1$.

Example

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(c) Linear k-step methods for which $\rho(z) = z^k - z^{k-2}$ are called **Nyström methods** if explicit and **Milne–Simpson methods** if implicit.

All these methods are zero-stable.

Consistency

Suppose that $y(x)$ is a solution of the ordinary differential equation $y' = f(x, y)$. Then the consistency error of a k-step method is:

$$T_n = \frac{\sum_{j=0}^{k} \left[\alpha_j y(x_{n+j}) - h\beta_j y'(x_{n+j})\right]}{h \sum_{j=0}^{k} \beta_j}. \tag{2}$$

The definition requires implicitly that $\sigma(1) = \sum_{j=0}^{k} \beta_j \neq 0$.

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Remark
As in the case of one-step methods, the consistency error can be thought of as the residual that is obtained by inserting the solution of the differential equation into the formula for the k-step method and scaling this residual appropriately (in this case dividing through by $h \sum_{j=0}^{k} \beta_j$) so that T_n resembles $y' - f(x, y(x))$.

Definition

A k-step method is said to be **consistent** with the differential equation $y' = f(x, y)$ if the consistency error defined by (2) is such that for any $\varepsilon > 0$ there exists an $h(\varepsilon)$ for which

$$|T_n| < \varepsilon \quad \text{for } 0 < h < h(\varepsilon),$$

and for any $(k+1)$ points $(x_n, y(x_n)), \ldots, (x_{n+k}, y(x_{n+k}))$ on any solution curve in R of $y' = f(x, y)$, $y(x_0) = y_0$.

Suppose that the exact solution y is sufficiently smooth, expand $y(x_{n+j})$ and $y'(x_{n+j})$ into a Taylor series about the point x_n and substitute these expansions into the numerator in (2). Thus,

$$T_n = \frac{1}{h\sigma(1)}[C_0 y(x_n) + C_1 hy'(x_n) + C_2 h^2 y''(x_n) + \cdots], \qquad (3)$$

where

$$\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j, \\ C_1 &= \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j, \\ &\text{etc.} \\ C_q &= \sum_{j=1}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=1}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j, \qquad \text{for } q \geq 2. \end{aligned}$$

For consistency we need that $T_n \to 0$ as $h \to 0$ and this requires that $C_0 = 0$ and $C_1 = 0$; in terms of the characteristic polynomials this consistency requirement can be restated in compact form as

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1) \neq 0.$$

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Observe that, according to this condition, if a linear multi-step method is consistent then it has a *simple* root on the unit circle at $z = 1$; thus the root condition is not violated by this zero.

Definition

A k-step method is said to have **order of consistency** p (or **order of accuracy** p) if p is the largest positive integer such that, for any sufficiently smooth solution curve in R of the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$, there exist constants K and h_0 such that

$$|T_n| \leq Kh^p \quad \text{for } 0 < h \leq h_0$$

for any $(k+1)$ points $(x_n, y(x_n)), \ldots, (x_{n+k}, y(x_{n+k}))$ on the solution curve.

Thus we deduce from (3) that the method is of order of consistency p if, and only if,

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In this case,

$$T_n = \frac{C_{p+1}}{\sigma(1)} h^p y^{(p+1)}(x_n) + \mathcal{O}(h^{p+1});$$

the number C_{p+1} ($\neq 0$) is called the **error constant** of the method.

Exercise

Construct an implicit linear two-step method of maximum order, containing one free parameter.

Determine the order and the error constant of the method.

SOLUTION: Taking $\alpha_0 = a$ as parameter, the method has the form

$$y_{n+2} + \alpha_1 y_{n+1} + a y_n = h(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n),$$

with $\alpha_2 = 1$, $\alpha_0 = a$, $\beta_2 \neq 0$.

SOLUTION: Taking $\alpha_0 = a$ as parameter, the method has the form

$$y_{n+2} + \alpha_1 y_{n+1} + a y_n = h(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n),$$

with $\alpha_2 = 1$, $\alpha_0 = a$, $\beta_2 \neq 0$. We have to determine α_1, β_2, β_1, β_0, so we need four equations; these will be arrived at by requiring that

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2, \\ C_1 &= \alpha_1 + 2 - (\beta_0 + \beta_1 + \beta_2), \\ C_q &= \frac{1}{q!}(\alpha_1 + 2^q \alpha_2) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2), \qquad q = 2, 3, \end{aligned}$$

appearing in (3) are all equal to zero, because we wish to maximise the order of the method.

SOLUTION: Taking $\alpha_0 = a$ as parameter, the method has the form

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appearing in (3) are all equal to zero, because we wish to maximise the order of the method. Thus,

$$\begin{aligned} C_0 &= a + \alpha_1 + 1 = 0, \\ C_1 &= \alpha_1 + 2 - (\beta_0 + \beta_1 + \beta_2) = 0, \\ C_2 &= \frac{1}{2!}(\alpha_1 + 4) - (\beta_1 + 2\beta_2) = 0, \\ C_3 &= \frac{1}{3!}(\alpha_1 + 8) - \frac{1}{2!}(\beta_1 + 4\beta_2) = 0. \end{aligned}$$

Hence,

$$\alpha_1 = -1 - a,$$
$$\beta_0 = -\frac{1}{12}(1 + 5a), \quad \beta_1 = \frac{2}{3}(1 - a), \quad \beta_2 = \frac{1}{12}(5 + a),$$

and the resulting method is

$$y_{n+2} - (1+a)y_{n+1} + ay_n = \frac{1}{12}h\left[(5 + a)f_{n+2} + 8(1 - a)f_{n+1} - (1 + 5a)f_n\right]. \tag{4}$$

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Further,

$$\begin{aligned} C_4 &= \frac{1}{4!}(\alpha_1 + 16) - \frac{1}{3!}(\beta_1 + 8\beta_2) = -\frac{1}{4!}(1+a), \\ C_5 &= \frac{1}{5!}(\alpha_1 + 32) - \frac{1}{4!}(\beta_1 + 16\beta_2) = -\frac{1}{3\cdot 5!}(17+13a). \end{aligned}$$

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If $a \neq -1$ then $C_4 \neq 0$, and the method (4) is third order consistent.

If, on the other hand, $a = -1$, then $C_4 = 0$ and $C_5 \neq 0$ and the method (4) becomes the Simpson rule method: a fourth-order consistent two-step method. The error constant is:

$$\begin{aligned} C_4 &= -\frac{1}{4!}(1+a), \qquad a \neq -1, \\ C_5 &= -\frac{4}{3 \cdot 5!}, \qquad a = -1. \end{aligned}$$

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