Numerical Solution of Differential Equations I

Endre Süli

Mathematical Institute
University of Oxford
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Lecture 7

Convergence

Linear k-step method:

$$\sum_{j=1}^{k} \alpha_j y_{n+j} = h \sum_{j=1}^{k} \beta_j f_{n+j}, \qquad n = 0, 1, \dots, N-k,$$

$h := (X_M - x_0)/N$, $N \gg 1$, $f_n := f(x_n, y_n)$, $\alpha_k \neq 0$, $\alpha_0^2 + \beta_0^2 \neq 0$.

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What matters most from the practical point of view is that the numerical approximation y_n at the mesh-point x_n is close to the value of the analytical solution $y(x_n)$, for $n = 0, \ldots, N$, and that the **global error** $e_n = y(x_n) - y_n$ tends to 0 when $h \to 0$.

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In order to formalise the desired behaviour, we introduce the following definition.

Definition

A linear multi-step method is said to be **convergent** if, for all initial-value problems $y' = f(x, y)$, $y(x_0) = y_0$, subject to the hypotheses of the Cauchy–Picard theorem, we have that

$$\lim_{\substack{h\to 0\\ nh=x-x_0}} y_n = y(x) \tag{1}$$

holds for all $x \in [x_0, X_M]$ and for all solutions $\{y_n\}_{n=0}^{N}$ generated by the k-step method with **consistent starting conditions**, i.e., with starting conditions $y_s = \eta_s(h)$, $s = 0, 1, \ldots, k-1$, for which $\lim_{h\to 0} \eta_s(h) = y_0$, $s = 0, 1, \ldots, k-1$.

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The key result is **Dahlquist's Equivalence Theorem**, which states that for a consistent linear multi-step method zero-stability is necessary and sufficient for convergence.

Necessary conditions for convergence

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Theorem

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Proof:

Suppose that a linear multi-step method is convergent; we wish to show that it is then zero-stable.

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We consider the initial-value problem $y' = 0$, $y(0) = 0$, on the interval $[0, X_M]$, $X_M > 0$, whose solution is, trivially, $y(x) \equiv 0$.

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Applying the method to this problem yields the difference equation

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = 0. \qquad (2)$$

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As the method is assumed to be convergent, for any $x \in [0, X_M]$ we have

$$\lim_{\substack{h \to 0 \\ nh = x}} y_n = 0, \qquad (3)$$

for all solutions of (2) s.t. $y_s = \eta_s(h)$, $s = 0, \ldots, k-1$, where

$$\lim_{h \to 0} \eta_s(h) = 0, \qquad s = 0, 1, \ldots, k-1. \qquad (4)$$

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This implies that $r \leq 1$. Thus we have proved that $|z| \leq 1$.

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As $y_n \to 0$ when $h \to 0$ and $n \to \infty$, the same must be true of the right-hand side. Thus, again, necessarily $r \leq 1$, i.e., $|z| \leq 1$.

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We shall prove below that this contradicts our assumption that the method (2) is convergent. It is easy to check that the numbers

$$y_n = h^{1/2} n r^n \cos n\phi \tag{5}$$

define a solution to (2). [Hint: $\mathrm{Re}\big(nz^n\rho(z) + z^{n+1}\rho'(z)\big) = 0$.]

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define a solution to (2). [Hint: $\mathrm{Re}\big(nz^n\rho(z) + z^{n+1}\rho'(z)\big) = 0$.]

In addition, (4) holds because

$$|\eta_s(h)| = |y_s| \le h^{1/2} s \le h^{1/2}(k-1), \qquad s = 0, \dots k-1.$$

CASE 2.1 If $\phi \neq 0$ and $\phi \neq \pi$, then

$$\frac{z_n^2 - z_{n+1} z_{n-1}}{\sin^2 \phi} = r^{2n}, \tag{6}$$

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Thus we have reached a contradiction.

CASE 2.2 If, on the other hand, $\phi = 0$ or $\phi = \pi$, it follows from (5) with $h = x/n$ that

$$|y_n| = x^{1/2} n^{1/2} r^n. \tag{7}$$

Since, by assumption, $|z| = 1$ (and therefore $r = 1$), we deduce from (7) that $\lim_{n\to\infty} |y_n| = \infty$, which again contradicts (3). $\diamond$

Theorem

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Remark. In other words, we need to show that if a linear multi-step method is convergent, then

$$C_0 = \sum_{j=0}^{k} \alpha_j = \rho(1) = 0,$$

and

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j = \rho'(1) - \sigma(1) = 0.$$

PROOF:

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We consider the initial-value problem $y' = 0$, $y(0) = 1$, on the interval $[0, X_M]$, $X_M > 0$, whose solution is, trivially, $y(x) \equiv 1$.

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Applying the method to this gives:

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We supply "exact" starting values for the numerical method; i.e., we choose $y_s = 1$, $s = 0, \ldots, k-1$. As, by hypothesis, the method is convergent, we deduce that

$$\lim_{\substack{h \to 0 \\ nh = x}} y_n = 1. \tag{9}$$

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By the definition of C_0, (11) is equivalent to $C_0 = 0$.

To show that $C_1 = 0$, we consider the initial-value problem $y' = 1$, $y(0) = 0$, on the interval $[0, X_M]$, $X_M > 0$; hence, $y(x) = x$.

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The method applied to this now becomes

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = h(\beta_k + \beta_{k-1} + \cdots + \beta_0), \quad (12)$$

where $X_M - x_0 = X_M - 0 = Nh$ and $1 \leq n \leq N - k$.

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where $X_M - x_0 = X_M - 0 = Nh$ and $1 \leq n \leq N - k$.

For a convergent method every solution of (12) satisfying

$$\lim_{h \to 0} \eta_s(h) = 0, \qquad s = 0, 1, \ldots, k-1, \quad (13)$$

where $y_s = \eta_s(h)$, $s = 0, 1, \ldots, k-1$, must also satisfy

$$\lim_{\substack{h \to 0 \\ nh = x}} y_n = x. \quad (14)$$

By the previous theorem zero-stability is necessary for convergence; so the first characteristic polynomial $\rho(z)$ of the method does not have a multiple root on the unit circle $|z| = 1$; therefore

$$\rho'(1) = k\alpha_k + \cdots + 2\alpha_2 + \alpha_1 \neq 0.$$

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Let the sequence $\{y_n\}_{n=0}^{N}$ be defined by $y_n = Knh$, where

$$K = \frac{\beta_k + \cdots + \beta_1 + \beta_0}{k\alpha_k + \cdots + 2\alpha_2 + \alpha_1} = \frac{\sigma(1)}{\rho'(1)}; \qquad (15)$$

this sequence clearly satisfies (13) and is the solution of (12).

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this sequence clearly satisfies (13) and is the solution of (12).

Furthermore, (14) implies that

$$x = y(x) = \lim_{\substack{h \to 0 \\ nh = x}} y_n = \lim_{\substack{h \to 0 \\ nh = x}} Knh = Kx,$$

and therefore $K = 1$. Hence, from (15),

$$C_1 = (k\alpha_k + \cdots + 2\alpha_2 + \alpha_1) - (\beta_k + \cdots + \beta_1 + \beta_0) = 0. \qquad \diamond$$

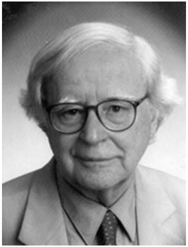
Sufficient conditions for convergence

Theorem

For a linear multi-step method that is consistent with the ordinary differential equation $y' = f(x, y)$*, where* f *is assumed to satisfy a Lipschitz condition, and starting with consistent starting conditions, zero-stability is sufficient for convergence.*

[Proof (optional): See the Lecture Notes.]

By combining the last three theorems we arrive at the following important result.



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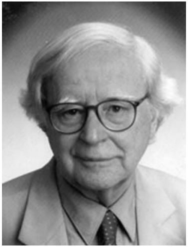

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Theorem (Dahlquist's Theorem)

For a linear multi-step method that is consistent with the ordinary differential equation $y' = f(x, y)$ *where* f *satisfies the Lipschitz condition, and starting with consistent initial data, zero-stability is necessary and sufficient for convergence. Moreover if the solution* $y(x)$ *has continuous derivative of order* $(p+1)$ *and consistency error* $\mathcal{O}(h^p)$*, then the global error* $e_n = y(x_n) - y_n$ *is also* $\mathcal{O}(h^p)$*.*

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By Dahlquist's theorem, if a linear multi-step method is not zero-stable then its global error cannot be made arbitrarily small by taking the mesh size h sufficiently small for any sufficiently accurate initial data.

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In fact, if the root condition is violated then there exists a solution to the linear multi-step method which will grow by an arbitrarily large factor in a fixed interval of x, however accurate the starting conditions are. This highlights the importance of zero-stability in practical computations.