Numerical Solution of Differential Equations I

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Lecture 9

Consider an initial-value problem for a *system* of m ODEs:

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{y}_0, \tag{1}$$

where $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_m)^{\mathrm{T}}$.

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A linear k-step method for the numerical solution of (1) is

$$\sum_{j=0}^{k} \alpha_j \mathbf{y}_{n+j} = h \sum_{j=0}^{k} \beta_j \mathbf{f}_{n+j}, \qquad \text{where } \mathbf{f}_{n+j} = \mathbf{f}(x_{n+j}, \mathbf{y}_{n+j}). \tag{2}$$

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Suppose, for simplicity, that $\mathbf{f}(x, \mathbf{y}) = A\mathbf{y} + \mathbf{b}$ where $A \in \mathbb{R}^{m \times m}$ is a constant matrix and $\mathbf{b} \in \mathbb{R}^m$ is a constant (column) vector.

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Then (2) becomes

$$\sum_{j=0}^{k} (\alpha_j I - h\beta_j A)\mathbf{y}_{n+j} = h\sigma(1)\mathbf{b}, \tag{3}$$

where $\sigma(1) = \sum_{j=0}^{k} \beta_j \, (\neq 0)$ and I is the $m \times m$ identity matrix.

Let us suppose that the eigenvalues λ_i, $i = 1, \ldots, m$, of the matrix A are distinct. Then, there exists a nonsingular matrix H such that

$$HAH^{-1} = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix}; \quad \text{i.e., } A = H^{-1}\Lambda H. \quad (4)$$

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$$\sum_{j=0}^{k} (\alpha_j I - h\beta_j \Lambda)\mathbf{z}_{n+j} = \mathbf{c}, \quad (5)$$

or, in component-wise form,

$$\sum_{j=0}^{k} (\alpha_j - h\beta_j \lambda_i) z_{n+j,i} = c_i,$$

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where $z_{n+j,i}$ and c_i, $i = 1, \ldots, m$, are the components of $\mathbf{z}_{n+j}$ and $\mathbf{c}$ respectively. Each of these m equations is completely decoupled from the other $m - 1$ equations.

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However, there is a new feature here: because the numbers λ_i, $i = 1, \ldots, m$, are eigenvalues of the matrix A, they need not be real numbers. As a consequence the parameter $\bar{h} := h\lambda$, where λ is any of the m eigenvalues, can be a complex number.

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This leads to the following modification of our earlier definition of absolute stability.

Definition

A linear k-step method is said to be **absolutely stable** in an open set $\mathcal{R}_A$ of the complex plane if, for all $\bar{h} \in \mathcal{R}_A$, all roots r_s, $s = 1, \ldots, k$, of the stability polynomial $\pi(r, \bar{h})$ associated with the method satisfy $|r_s| < 1$. The set $\mathcal{R}_A$ is called the **region of absolute stability** of the method.

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Clearly, the interval of absolute stability of a linear multistep method is a subset of its region of absolute stability.

Exercise

a) *Find the region of absolute stability of Euler's explicit method when applied to* $y' = \lambda y$, $y(x_0) = y_0$, $\lambda \in \mathbb{C}$, $\mathrm{Re}\ \lambda < 0$.

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b) *Suppose that Euler's explicit method is applied to the second-order differential equation*

$$y'' + (1-\lambda)y' - \lambda y = 0, \qquad y(0) = 1, \quad y'(0) = -\lambda - 2,$$

rewritten as a first-order system in the vector $(u, v)^{\mathrm{T}}$, *with* $u = y$ *and* $v = y'$, $\lambda \in \mathbb{C}$, $\mathrm{Re}\ \lambda < 0$, *and let* $|\lambda| \gg 1$.

What choice of the step size $h \in (0, 1)$ *will guarantee absolute stability in the sense of the last definition?*

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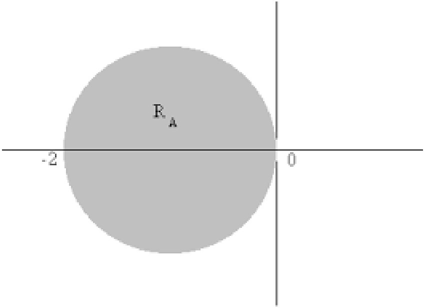
This has the root $r = 1 + \bar{h}$. Hence the region of absolute stability is

$$\mathcal{R}_A = \{\bar{h} \in \mathbf{C} \,:\, |1+\bar{h}| < 1\},$$

which is an open unit disc centred at -1.



b) Now writing $u = y$ and $v = y'$ and $\mathbf{y} = (u, v)^{\mathrm{T}}$, the initial-value problem for the given second-order differential equation can be recast as

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whose roots are -1 and λ, and we deduce that the method is absolutely stable provided that $|1 + h\lambda| < 1$. It is an easy matter to show that

$$u(x) = 2\mathrm{e}^{-x} - \mathrm{e}^{\lambda x}, \qquad v(x) = -2\mathrm{e}^{-x} + \lambda \mathrm{e}^{\lambda x}.$$

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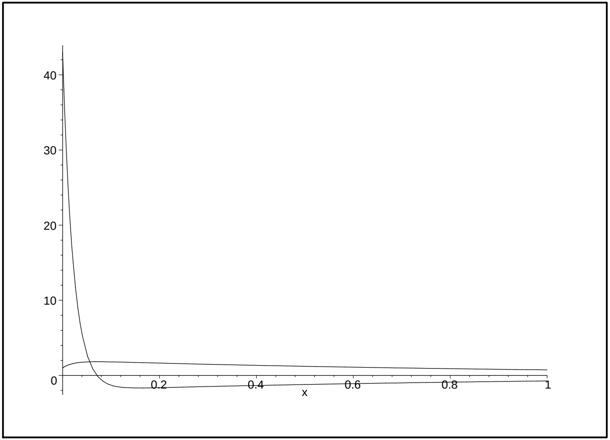
The graphs of u and v are shown on the next slide for $\lambda = -45$. $\diamond$



v varies rapidly near $x = 0$ while u is slowly varying for $x > 0$ and v is slowly varying for $x > 1/45$. Nevertheless, we are forced to use a step size of $h < 2/45$ in order to ensure that the method is absolutely stable.

To ensure the absolute stability, the mesh size *h* may have to be chosen exceedingly small, $h < -2\mathrm{Re}\ \lambda/|\lambda|^2$, smaller than an accurate approximation of the solution for $x \gg 1/|\lambda|$ would necessitate. Systems of differential equations which exhibit this behaviour are generally referred to as **stiff systems**.

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A historic and pragmatic 'definition' by Curtis and Hirschfelder[2] reads: stiff equations are equations where the implicit Euler method works significantly better than the explicit Euler method.

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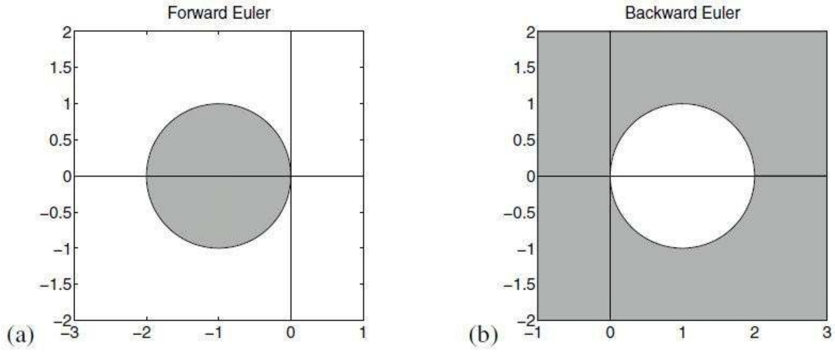
Since the only root of the stability polynomial is $z = 1/(1 - \bar{h})$, we deduce that the method has the region of absolute stability

$$\mathcal{R}_A = \{\bar{h} \in \mathbb{C} \,:\, |1 - \bar{h}| > 1\}.$$

$\mathcal{R}_A$ includes the whole of the left open complex half-plane.



The shaded regions are the regions of absolute stability of the explicit (forward) Euler and the implicit (backward) Euler method in the complex plane.

Definition (Dahlquist (1963))

A linear multistep method is said to be A-stable if its region of absolute stability, $\mathcal{R}_A$, contains the whole of the left open complex half-plane $\mathrm{Re}(h\lambda) < 0$.

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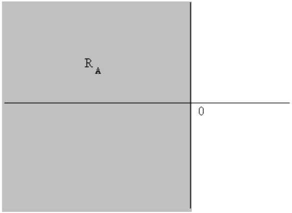

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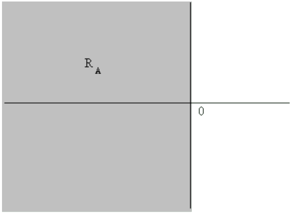

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As the next theorem shows, this definition is far too restrictive.

Theorem (Dahlquist (1963))

(i) *No* explicit *linear multistep method is A-stable.*

(ii) *The order of an A-stable* implicit *linear multistep method cannot exceed* 2*.*

(iii) *The second-order A-stable linear multistep method with smallest error constant is the trapezium rule.*

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This motivates the following, less restrictive notion of stability.

Definition (Widlund (1967))

A linear multistep method is said to be $A(\alpha)$-**stable**, $\alpha \in (0, \pi/2)$, if its region of absolute stability $\mathcal{R}_A$ contains the infinite open wedge in the complex plane

$$W_\alpha = \{\bar{h} \in \mathbb{C} \,|\, \pi - \alpha < arg(\bar{h}) < \pi + \alpha\}.$$

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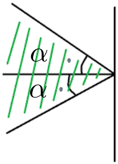

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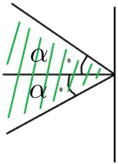
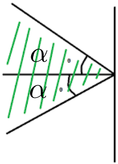


A linear multistep method is A_0 stable if $\mathcal{R}_A$ includes the negative real axis in the complex plane.



Remark

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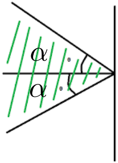

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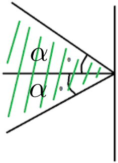

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In particular, if all eigenvalues of A *are real and negative, then an* $A(0)$ *stable method can be used.*

Theorem

(i) *No* explicit *linear multistep method is* $A(0)$*-stable.*

(ii) *The only* $A(0)$*-stable linear* k*-step method whose order exceeds* k *is the trapezium rule.*

(iii) *For each* $\alpha \in [0, \pi/2)$ *there exist* $A(\alpha)$*-stable linear* k*-step methods of order* p *for which* $k = p = 3$ *and* $k = p = 4$*.*

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The motivation behind it is the fact that for a typical stiff problem the eigenvalues of the matrix A which produce the fast transients all lie to the left of a line $\mathrm{Re}\,\bar{h} = -a$, $a > 0$, in the complex plane, while those that are responsible for the slow transients are clustered around zero.

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Definition (Gear (1969))

A linear multistep method is said to be **stiffly stable** if there exist positive real numbers a and c such that $\mathcal{R}_A \supset \mathcal{R}_1 \cup \mathcal{R}_2$ where

$$\mathcal{R}_1 = \{\bar{h} \in \mathbf{C} \,:\, \mathrm{Re}\,\bar{h} < -a\},$$
$$\mathcal{R}_2 = \{\bar{h} \in \mathbf{C} \,:\, -a \le \mathrm{Re}\,\bar{h} < 0, \;\; -c \le \mathrm{Im}\,\bar{h} \le c\}.$$

It is clear that stiff stability implies $A(\alpha)$-stability with

$$\alpha = \arctan(c/a).$$

More generally, we have the following chain of implications:

A-stability $\Rightarrow$ stiff-stability $\Rightarrow$ $A(\alpha)$-stability $\Rightarrow$ $A(0)$-stability $\Rightarrow$ A_0-stability.