Numerical Solution of Differential Equations I

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2020

Lecture 12

Finite difference approximation of parabolic equations

As a simple but representative model problem we focus on the unsteady diffusion equation (heat equation) in one space dimension:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

which we shall consider for $x \in (-\infty, \infty)$ and $t \geq 0$, subject to the initial condition

$$u(x, 0) = u_0(x), \qquad x \in (-\infty, \infty),$$

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We summarize here the derivation of this expression.

We recall that the Fourier transform of a function v is defined by

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By Fourier-transforming the PDE (1) we obtain

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\,\mathrm{e}^{-\imath x\xi}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x,t)\,\mathrm{e}^{-\imath x\xi}\,\mathrm{d}x.$$

After (formal) integration by parts on the right-hand side and ignoring boundary terms at $\pm\infty$, we obtain

$$\frac{\partial}{\partial t}\hat{u}(\xi, t) = (\imath\xi)^2\hat{u}(\xi, t),$$

whereby

$$\hat{u}(\xi, t) = \mathrm{e}^{-t\xi^2}\hat{u}(\xi, 0),$$

and therefore

$$u(x, t) = F^{-1}\left(\mathrm{e}^{-t\xi^2}\hat{u}_0\right).$$

The inverse Fourier transform of a function is defined by

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After some lengthy calculations, which we omit, we find that

$$u(x,t) = F^{-1}\left(\mathrm{e}^{-t\xi^2}\hat{u}_0(\xi)\right) = \int_{-\infty}^{\infty} w(x-y,t)u_0(y)\,\mathrm{d}y,$$

where the function w, defined by

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is called the **heat kernel**. So, finally,

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-(x-y)^2/(4t)} u_0(y) \, \mathrm{d}y. \tag{2}$$

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$$\int_{-\infty}^{\infty} w(y,t)\,\mathrm{d}y = 1 \qquad \text{for all } t > 0,$$

we deduce from (2) that if u_0 is a bounded continuous function, then

$$\sup_{x\in(-\infty,+\infty)}|u(x,t)| \leq \sup_{x\in(-\infty,\infty)}|u_0(x)|, \qquad t > 0. \tag{3}$$

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In other words, the 'largest' and 'smallest' values of $u(\cdot,t)$ at $t > 0$ cannot exceed those of $u_0(\cdot)$.

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We will show, using Parseval's identity, that the L^2 norm of the solution, at any time $t > 0$, is bounded by the L^2 norm of the initial datum.

We shall then try to mimic this when using various numerical approximations of the initial-value problem for the heat equation.

Lemma (Parseval's identity)

Suppose that $u \in L^2(-\infty, \infty)$*. Then,* $\hat{u} \in L^2(-\infty, \infty)$*, and the following equality holds:*

$$\|u\|_{L^2(-\infty,\infty)} = \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L^2(-\infty,\infty)},$$

where

$$\|u\|_{L^2(-\infty,\infty)} = \left(\int_{-\infty}^{\infty} |u(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

Proof. We begin by observing that

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{u}(\xi)\, v(\xi)\, \mathrm{d}\xi &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(x)\, \mathrm{e}^{-\imath x \xi}\, \mathrm{d}x \right) v(\xi)\, \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(\xi)\, \mathrm{e}^{-\imath x \xi}\, \mathrm{d}\xi \right) u(x)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} u(x)\, \hat{v}(x)\, \mathrm{d}x.\end{aligned}$$

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We then take

$$v(\xi) = \overline{\hat{u}(\xi)} = 2\pi F^{-1}[\bar{u}](\xi)$$

and substitute this into the identity above. $\diamond$

Returning to equation (1), we thus have by Parseval's identity that

$$\|u(\cdot,t)\|_{L^2(-\infty,\infty)} = \frac{1}{\sqrt{2\pi}} \|\hat{u}(\cdot,t)\|_{L^2(-\infty,\infty)}, \qquad t > 0.$$

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Therefore,

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2(-\infty,\infty)} &= \frac{1}{\sqrt{2\pi}}\,\|\mathrm{e}^{-t\xi^2}\hat{u}_0(\cdot)\|_{L^2(-\infty,\infty)} \\
&\leq \frac{1}{\sqrt{2\pi}}\,\|\hat{u}_0\|_{L^2(-\infty,\infty)} \\
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Thus we have shown that

$$\|u(\cdot,t)\|_{L^2(-\infty,\infty)} \le \|u_0\|_{L^2(-\infty,\infty)} \qquad \text{for all } t > 0. \tag{4}$$

This is a useful result as it can be used to deduce stability of the solution of the equation (1) with respect to perturbations of the initial datum in a sense which we shall now explain.

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Suppose that u_0 and $\tilde{u}_0$ are two functions contained in $L^2(-\infty, \infty)$ and denote by u and $\tilde{u}$ the solutions to (1) resulting from the initial functions u_0 and $\tilde{u}_0$, respectively.

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Then $u - \tilde{u}$ solves the heat equation with initial datum $u_0 - \tilde{u}_0$, and therefore, by (4), we have that

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^2(-\infty,\infty)} \leq \|u_0 - \tilde{u}_0\|_{L^2(-\infty,\infty)} \qquad \text{for all } t > 0.$$

This inequality implies continuous dependence of the solution on the initial function: small perturbations in u_0 in the $L^2(-\infty, \infty)$ norm will result in small perturbations in the associated analytical solution $u(\cdot, t)$ in the $L^2(-\infty, \infty)$ norm for all $t > 0$.

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Analogously,

$$\sup_{x \in (-\infty, \infty)} |u(x,t) - \tilde{u}(x,t)| \leq \sup_{x \in (-\infty, \infty)} |u_0(x) - \tilde{u}_0(x)| \qquad \text{for all } t > 0.$$