Numerical Solution of Differential Equations I

Endre Süli

Mathematical Institute
University of Oxford
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Lecture 15

Boundary-value problems for parabolic problems

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Consider the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad a < x < b, \quad 0 < t \leq T,$$

subject to the initial condition

$$u(x, 0) = u_0(x), \qquad x \in [a, b],$$

and the Dirichlet boundary conditions at $x = a$ and $x = b$:

$$u(a, t) = A(t), \quad u(b, t) = B(t), \qquad t \in (0, T].$$

Remark

The Neumann initial-boundary-value problem for the heat equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad a < x < b, \quad 0 < t \leq T,$$

subject to the initial condition

$$u(x,0) = u_0(x), \qquad x \in [a,b],$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(a,t) = A(t), \quad \frac{\partial u}{\partial x}(b,t) = B(t), \qquad t \in (0,T].$$

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Let $\Delta x = (b-a)/J$ and $\Delta t = T/M$, and define

$$x_j := a + j\Delta x, \quad j = 0, \ldots, J, \qquad t_m := m\Delta t, \quad m = 0, \ldots, M.$$

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$$x_j := a + j\Delta x, \quad j = 0, \ldots, J, \qquad t_m := m\Delta t, \quad m = 0, \ldots, M.$$

We approximate the Dirichlet initial-boundary-value problem with the θ-scheme:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\,\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} + \theta\,\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},$$

for $j = 1, \ldots, J-1$, $m = 0, 1, \ldots, M-1$,

$$U_j^0 = u_0(x_j), \qquad j = 1, \ldots, J-1,$$

$$U_0^{m+1} = A(t_{m+1}), \;\; U_J^{m+1} = B(t_{m+1}), \;\; m = 0, \ldots, M-1.$$

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$$\begin{aligned} [1-\theta\mu\delta^2]U_j^{m+1} &= [1+(1-\theta)\mu\delta^2]U_j^m, \\ U_j^0 &= u_0(x_j), \qquad 1 \le j \le J-1, \end{aligned}$$

$$U_0^{m+1} = A(t_{m+1}), \qquad U_J^{m+1} = B(t_{m+1}), \;\; 0 \le m \le M-1,$$

where

$$\delta^2 U_j := U_{j+1} - 2U_j + U_{j-1}.$$

Consider the symmetric tridiagonal $(J-1) \times (J-1)$ matrix:

$$\mathcal{A} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

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Let $\mathcal{I} = \mathrm{diag}(1,\, 1,\, 1, \dots,\, 1,\, 1)$ be the $(J-1) \times (J-1)$ identity matrix.

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Let $\mathcal{I} = \mathrm{diag}(1, 1, 1, \dots, 1, 1)$ be the $(J-1) \times (J-1)$ identity matrix. Then, the θ-scheme can be written as

$$(\mathcal{I} - \theta\mu\mathcal{A})\mathbf{U}^{m+1} = (\mathcal{I} + (1-\theta)\mu\mathcal{A})\mathbf{U}^m + \theta\mu\mathbf{F}^{m+1} + (1-\theta)\mu\mathbf{F}^m$$

for $m = 0, 1, \dots, M-1$, where

$$\mathbf{U}^m = (U_1^m,\ U_2^m,\ \dots,\ U_{J-2}^m,\ U_{J-1}^m)^{\mathrm{T}}$$

and

$$\mathbf{F}^m = (A(t_m),\ 0,\ \dots,\ 0,\ B(t_m))^{\mathrm{T}}.$$