Further Mathematical Biology: Problem Sheet 3

Michaelmas Term 2020

Question 1.

Suppose that a population $u(x,t)$ satisfies the one-dimensional reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + g(u), \tag{1}$$

where the diffusion coefficient is $D(u) > 0$, and the net production rate is $g(u)$ where $g(0) = g(1) = 0$, $g(u) > 0$, $\forall u \in (0,1)$, $g'(0) > 0$, $g'(1) < 0$, and both g and D are continuously differentiable.

(a) Show that if a travelling wave, $u = U(z)$ where $z = x - ct$, exists from $u = 1$ to $u = 0$, then

$$c = \frac{\displaystyle\int_0^1 g(w)D(w)\mathrm{d}w}{\displaystyle\int_{-\infty}^{\infty} D(U(s))\left[\frac{\mathrm{d}U(s)}{\mathrm{d}s}\right]^2 \mathrm{d}s}, \tag{2}$$

and hence that $c > 0$.

[Hint: Convert to travelling wave coordinates, then multiply by DU' and integrate.]

(b) Assuming that such a travelling wave solution is possible, find the lower limit on the wave speed.

(c) Sketch $U(z)$ for the travelling wave solution, with the direction of motion clearly marked.

Question 2.

Consider a population of cells $u = u(x,t)$ that undergoes logistic growth but whose diffusion depends linearly on their density so that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u(1-u). \tag{3}$$

(a) Seek travelling wave solutions of the form $u(x,t) = U(z)$ where $z = x - ct$ and $c > 0$. Determine the equations satisfied by $U(z)$ and $V(z) = U'(z)$.

(b) By looking for a solution of the form $V = c(U-1)$, show that there is an exact solution for one value of the wave speed, c. You should state this value of c.

Question 3.

A non-dimensionalised version of the red-grey squirrel competition model takes the form

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} + G(1 - G - \gamma_1 R), \tag{4}$$

$$\frac{\partial R}{\partial t} = D\frac{\partial^2 R}{\partial x^2} + \alpha R(1 - R - \gamma_2 G), \tag{5}$$

where $G(x,t)$ and $R(x,t)$ are the densities of grey and red squirrels, respectively, at time t and position x, and D, α, γ_1 and γ_2 are positive constants with $\gamma_1 < 1$ and $\gamma_2 > 1$.

We seek travelling wave solutions of the form $G = G(z)$, $R = R(z)$ where $z = x - ct$ and c is a positive constant, with $G(-\infty) = 1$, $R(-\infty) = 0$, $G(\infty) = 0$ and $R(\infty) = 1$.

(a) Write down the system in travelling wave coordinates.

(b) In the special case where $D = 1$, $\alpha = 1$, $\gamma_1 + \gamma_2 = 2$, show that $S = G + R$ satisfies the equation

$$\frac{\mathrm{d}^2 S}{\mathrm{d}z^2} + c\frac{\mathrm{d}S}{\mathrm{d}z} + S(1 - S) = 0, \tag{6}$$

and hence, by considering boundary conditions, that $S \equiv 1$ for all z is a solution.

(c) Deduce that, for this special case,

$$\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} + c\frac{\mathrm{d}G}{\mathrm{d}z} + (1 - \gamma_1)G(1 - G) = 0. \tag{7}$$

(d) Show that, for this equation, travelling waves are possible if $c \geq 2(1 - \gamma_1)^{1/2}$ and sketch the wave.

Question 4.

A rabies model which includes logistic growth for the susceptibles, $S(x, t)$, and diffusive dispersal for the infectives, $I(x, t)$, is

$$\frac{\partial I}{\partial t} = D\frac{\partial^2 I}{\partial x^2} + rIS - aI, \tag{8}$$

$$\frac{\partial S}{\partial t} = -rIS + BS\left(1 - \frac{S}{S_0}\right), \tag{9}$$

where r, a, B, D and s_0 are positive parameters.

(a) Non-dimensionalise the system to give

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2} + uv - \lambda u, \tag{10}$$

$$\frac{\partial v}{\partial \tau} = -uv + bv(1 - v), \tag{11}$$

where u relates to I and v to S.

(b) Look for travelling wave solutions of the dimensionless equations from (a) with $u > 0$ and $v > 0$. By linearising far ahead of the wavefront, where the population is still fully susceptible and the infection has not yet arrived (*i.e.* $v \to 1$ and $u \to 0$), show that a wave may exist if $0 < \lambda < 1$ and, in this case, the wave speed is such that $c \geq 2\sqrt{(1 - \lambda)}$.