Further Mathematical Biology: Problem Sheet 4

Michaelmas Term 2020

Question 1.

Consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + f(u,v), \tag{1}$$

$$\frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + g(u,v), \tag{2}$$

where f and g describe the reaction kinetics, and D_1 and D_2 are positive constants.

(a) Derive the conditions for diffusion-driven instability.

(b) Show that, when these conditions hold, bifurcation to solutions oscillating in time (and space) cannot occur.

Question 2.

Consider the Gierer-Meinhardt reaction-diffusion system in one spatial dimension

$$\frac{\partial A}{\partial t} = D_A \frac{\partial^2 A}{\partial x^2} + \frac{\rho A^2}{(1+KA^2)H} - \mu A, \tag{3}$$

$$\frac{\partial H}{\partial t} = D_H \frac{\partial^2 H}{\partial x^2} + \rho' A^2 - \nu H, \tag{4}$$

where A and H are the reactant concentrations and ρ, K, μ, ν, ρ', D_A and D_H are positive constants.

(a) Draw a phase potrait of the system in the absence of diffusion and show that a diffusion-driven instability may be possible if the null clines intersect in a certain way.

(b) Write down the conditions for diffusion-driven instability.

[In (a) and (b) consider only the non-zero steady states.]

Question 3.

The amoebae of the slime mould *Dictyostelium discoideum* secrete a chemical attractant, cyclic-AMP, and spatial aggregations of the amoebae start to form. This process can be modelled by the following system of dimensional equations

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} = \tilde{D}_n \frac{\partial^2 \tilde{n}}{\partial \tilde{x}^2} - \tilde{\chi} \frac{\partial}{\partial \tilde{x}} \left(\tilde{n} \frac{\partial \tilde{a}}{\partial \tilde{x}} \right), \tag{5}$$

$$\frac{\partial \tilde{a}}{\partial \tilde{t}} = \tilde{D}_a \frac{\partial^2 \tilde{a}}{\partial \tilde{x}^2} + h\tilde{n} - k\tilde{a}, \tag{6}$$

where $\tilde{n}(\tilde{x},\tilde{t})$ and $\tilde{a}(\tilde{x},\tilde{t})$ are the cell density of the amoebae and the attractant concentration, respectively. The parameters h, k, $\tilde{\chi}$ and the diffusion coefficients, $\tilde{D}_n$ and $\tilde{D}_a$, are positive constants.

(a) Non-dimensionalise the system to obtain

$$\frac{\partial n}{\partial t} = D_n \frac{\partial^2 n}{\partial x^2} - \chi \frac{\partial}{\partial x}\left(n\frac{\partial a}{\partial x}\right), \tag{7}$$

$$\frac{\partial a}{\partial t} = D_a \frac{\partial^2 a}{\partial x^2} + n - a, \tag{8}$$

where the variables and parameters are now dimensionless.

(b) Suppose that the amoebae and chemical occupy an infinite domain. Examine the linear stability about the spatially uniform steady state $n = 1 = a$, and derive the dispersion relation. Obtain the conditions on the parameters for the mechanism to initiate spatially heterogeneous solutions.

(c) Suppose now that the amoebae and the chemical attractant are confined within a finite domain ($0 \leq \tilde{x} \leq L$ in dimensional variables), with zero flux boundary conditions imposed on both n and a at the ends of the domain. Determine the minimum domain size for which spatially structured solutions can arise.

(d) Briefly describe the physical processes operating and explain intuitively how spatial aggregation takes place.

Question 4.

Consider a tissue containing cells of density $n(x,t)$ which produce a chemical $c(x,t)$ according to the model equations

$$\frac{\partial n}{\partial t} = \mu \frac{\partial^2 n}{\partial x^2} - \frac{\partial}{\partial x}\left(n\frac{\partial c}{\partial x}\right), \tag{9}$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + \frac{n}{(1+n)^2} - c, \tag{10}$$

where μ and D are positive constants.

(a) If n_0 is the initial, spatially-uniform distribution of cells, what is the corresponding initial distribution of chemoattractant (*i.e.* what is the steady state for c when $n = n_0$)?

(b) Carry out a linear stability analysis about the positive steady state by seeking solutions of the form

$$(n, c) \sim (n_0, c_0) + e^{ikx+\sigma t}(N, K), \qquad 0 < |N|, |K| \ll n_0, \tag{11}$$

and obtaining a dispersion relation for σ in terms of the wavenumber, k.

(c) Let $\mu_{\max} = \max_x[x(1-x)/(1+x)^3]$. Show that when $0 < \mu < \mu_{\max}$, no spatial patterns can be obtained if the initial cell density, n_0, is sufficiently large or sufficiently small.